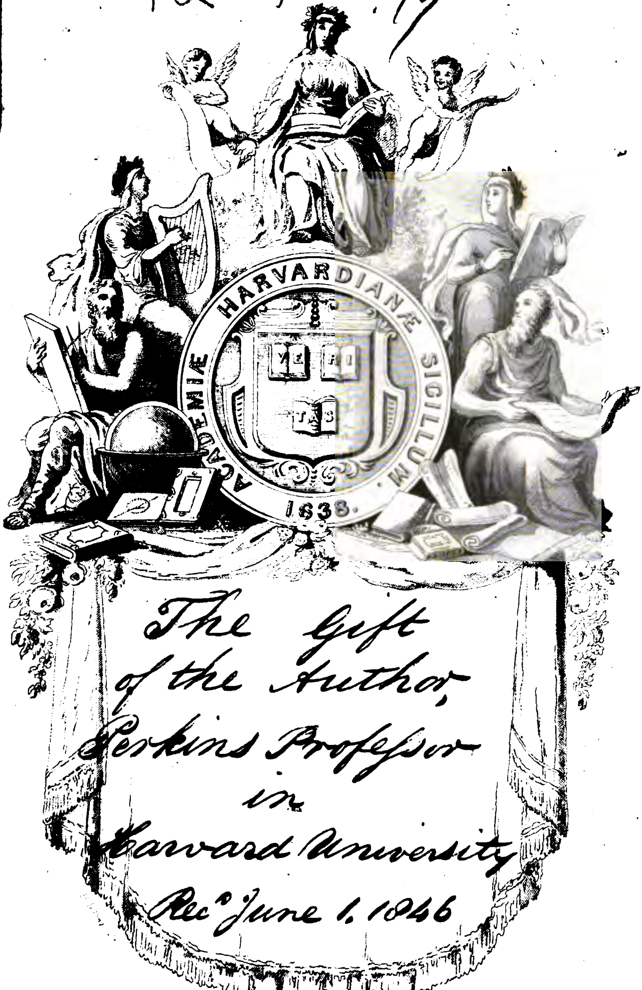
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AN

ELEMENTARY TREATISE

ON

CURVES, FUNCTIONS, AND FORCES.

VOLUME SECOND;

CONTAINING

CALCULUS OF IMAGINARY QUANTITIES, RESIDUAL CALCULUS, AND INTEGRAL CALCULUS.

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"	(140)	$2\,B$	B.
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BOOK III.

CALCULUS OF IMAGINARY QUANTITIES.

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CALCULUS OF IMAGINARY QUANTITIES.

CHAPTER I.

MODULUS AND ARGUMENT.

1. THE *general form*, to which geometers attempt to reduce all imaginary expressions, is that of *a binomial, in which one term is real and the other term is the product of a real factor by the imaginary factor* $\sqrt{-1}$.

a. Thus, if A denote the real term, and B the real factor of the imaginary term, this binomial type of imaginary quantities is

$$A + B\sqrt{-1}. \tag{1}$$

b. As this expression is imaginary, all operations, such as addition, multiplication, &c., performed upon it or by it, are wholly devoid of their usual meaning, and may admit of any conventional interpretation. But, then, rules must be adopted for performing the operations which shall be consistent with this interpretation; or, reciprocally, the rules for performing the operations may be assumed at pleasure, provided that a

mode of interpreting the operations and the results is adopted, which is consistent with the rules. Now, if we take

$$m^2 = -1, \quad \text{that is,} \quad m = \sqrt{-1}; \tag{2}$$

(1) becomes

$$A + Bm; \tag{3}$$

and all algebraical operations may be performed according to the usual rules upon (3), without any regard to the imaginary value of m, provided that the results are interpreted consistently with this imaginary value of m, and the real value of m^2, which is -1.

2. *The modulus of an imaginary expression of the form* (1) *is the positive square root of the sum of the squares of its real term, and of the real factor of its imaginary term.*

The argument of this imaginary expression is the angle, whose tangent is equal to the quotient of the real factor of its imaginary term divided by the real term.

Thus, if R denotes the modulus of (1), and θ its argument, we have

$$R = \sqrt{(A^2 + B^2)}, \tag{4}$$

$$\text{tang.}\ \theta = \frac{B}{A}. \tag{5}$$

3. *Corollary.* If A and B are represented by the sides of a right triangle, R is the hypothenuse, and θ is the angle opposite to B. Hence, by trigonometry,

$$A = R \cos.\ \theta, \tag{6}$$

$$B = R \sin.\ \theta; \tag{7}$$

and the value of (1) becomes

$$R\,(\cos.\,\theta + \sin.\,\theta.\,\sqrt{-1}). \tag{8}$$

4. *Corollary.* Since two angles, which differ by two right angles, have the same tangent, there are two values of θ less than four right angles, which satisfy (5); and of these two values, that one is to be selected which agrees, in the signs of its sine and cosine, with (6) and (7). Any angle, which differs from the value of θ thus found by four right angles, or by any multiple of four right angles, may also be taken as a value of θ. Thus, if θ_0 is this least positive value of θ, the general value of θ is

$$\theta = \theta_0 \pm 2\,n\,\pi, \tag{9}$$

in which n is any integer, and π is the ratio of the circumference to the diameter.

5. *Corollary.* When the imaginary part of (1) vanishes, we have

$$B = 0, \quad \sin.\,\theta = 0; \tag{10}$$

so that $\theta_0 = 0, \quad \theta = \pm 2\,n\,\pi, \qquad \cos.\,\theta = 1;$ (11)

or $\theta_0 = \pi, \quad \theta = \pm (2\,n + 1)\,\pi, \quad \cos.\,\theta = -1.$ (12)

and (11) corresponds to the case of *a real positive quantity*, (12) to that of *a real negative quantity.*

6. *Corollary.* When the real term of (1) vanishes, we have

$$A = 0, \quad \cos.\,\theta = 0, \quad \theta_0 = \tfrac{1}{2}\,\pi \text{ or } = \tfrac{3}{2}\,\pi, \tag{13}$$

whence

$$\theta = \pm \tfrac{1}{2}\,\pi \pm 2\,n\,\pi = \pm (2\,n \pm \tfrac{1}{2})\pi, \; \sin.\,\theta = \pm 1. \tag{14}$$

7. *Theorem.* When the quantity represented by (1) vanishes, the real and the imaginary part, and the modulus, are each equal to zero, while the argument is indeterminate.

Proof. For if

$$A + B\sqrt{-1} = 0, \tag{15}$$

we have
$$A = -B\sqrt{-1}; \tag{16}$$

that is, a real quantity equal to an imaginary one, which is impossible, and (16) cannot be satisfied, unless we have

$$A = 0, \quad B = 0; \tag{17}$$

whence, by (4),
$$R = 0, \tag{18}$$

and, by (5), θ is indeterminate.

8. *Theorem. When two imaginary quantities are equal, their real and imaginary parts are separately equal, and they have the same modulus and argument.*

Proof. For the equation

$$A + B\sqrt{-1} = A' + B'\sqrt{-1}, \tag{19}$$

gives, by transposition,

$$A - A' + (B - B')\sqrt{-1} = 0. \tag{20}$$

Hence, by the preceding theorem,

$$A - A' = 0, \quad B - B' = 0;$$

or,
$$A = A', \qquad B = B'; \tag{21}$$

whence, by (4 and 5),

$$R = R', \qquad \theta = \theta'. \tag{22}$$

9. Two imaginary quantities are *conjugate to each other*, when they have the same modulus, and when their arguments only differ in being of contrary signs.

Thus the conjugate of (8) is

$$R\,[\cos.(-\theta) + \sin.(-\theta)\,.\sqrt{-1}]; \tag{23}$$

or, by trigonometry,

$$R\,(\cos.\theta - \sin.\theta\,.\sqrt{-1}). \tag{24}$$

10. *Corollary.* Two imaginary quantities, which are conjugate to each other, differ only in the sign which precedes the imaginary part.

Thus $A + B\sqrt{-1}$ and $A - B\sqrt{-1}$ are, by (8 and 24), conjugate to each other.

11. *Theorem. The modulus of the product of several imaginary quantities is equal to the product of the moduli of the factors, and the argument of the product is equal to the sum of the arguments of the factors.*

Proof. *a.* When there are two factors

$$R\,(\cos.\theta + \sin.\theta.\sqrt{-1}) \text{ and } R'(\cos.\theta' + \sin.\theta'.\sqrt{-1}), \tag{25}$$

the product is

$$\begin{aligned} R\,R'\,&[\cos.\theta\cos.\theta' - \sin.\theta\sin.\theta'] \\ &+ (\sin.\theta\cos.\theta' + \sin.\theta'\cos.\theta)\sqrt{-1}, \end{aligned} \tag{26}$$

which, by (26 and 28 of Trig.), becomes

$$R\,R'\,[\cos.(\theta + \theta') + \sin.(\theta + \theta')\,.\sqrt{-1}]; \tag{27}$$

so that its modulus is the product of the two moduli, and its argument is the sum of the two arguments.

Imaginary power.

b. A third factor might be multiplied by (27) in the same way, that is, by multiplying its modulus by the new modulus, and adding to its argument the new argument; and this process might be extended to any number of factors.

12. *Corollary.* If the factors are all equal, the product becomes a power; whence the *modulus of a positive integral power of an imaginary quantity is the same power of its modulus, and the argument of the power is the product of its argument by the exponent of the power.*

Thus

$$[R(\cos.\theta+\sin.\theta.\sqrt{-1})]^n = R^n(\cos.n\theta+\sin.n\theta.\sqrt{-1}). \quad (28)$$

13. *Corollary.* When $R = 1$, (29)

(28) becomes

$$(\cos.\theta+\sin.\theta.\sqrt{-1})^n = (\cos.n\theta+\sin.n\theta.\sqrt{-1}). \quad (30)$$

Reversing the sign of θ

$$(\cos.\theta-\sin.\theta.\sqrt{-1})^n = (\cos.n\theta-\sin.n\theta.\sqrt{-1}). \quad (31)$$

14. *Corollary.* Half the sum of (30 and 31) is

$$\cos.n\theta = \tfrac{1}{2}(\cos.\theta+\sin.\theta.\sqrt{-1})^n + \tfrac{1}{2}(\cos.\theta-\sin.\theta.\sqrt{-1})^n. \quad (32)$$

Half the difference of (30 and 31) is

$$\sin.n\theta.\sqrt{-1} = \tfrac{1}{2}(\cos.\theta.+\sin.\theta.\sqrt{-1})^n - \tfrac{1}{2}(\cos.\theta-\sin.\theta.\sqrt{-1})^n \quad (33)$$

15. *Corollary.* By development, (32) becomes

$$\cos. n\,\theta = \cos.^n\theta - \frac{n\,(n-1)}{1\,.\,2}\cos.^{n-2}\theta\,\sin.^2\theta$$

$$+\frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\;2\,.\;3\,.\;4}\cos.^{n-4}\theta\,\sin.^4\theta - \&c. \quad (34)$$

By developing and dividing by $\sqrt{-1}$, (33) becomes

$$\sin.\,n\,\theta = n\cos.^{n-1}\theta\,\sin.\,\theta$$

$$-\frac{n\,(n-1)\,(n-2)}{1\,.\;2\,.\;3}\cos.^{n-3}\theta\,\sin.^3\theta + \&c. \quad (35)$$

16. *Corollary.* The reverse of § 12 is, that the modulus of a positive integral root of an imaginary quantity is the same root of its modulus, and the argument of the root is the quotient of its argument divided by the exponent of the root; that is, since roots are fractional powers, *the rule of* § 12 *extends to the case of positive fractional powers.*

17. *Corollary.* The product of two conjugate factors is equal to the square of the modulus.

For, in this case, (23 and 27) give

$$\theta + \theta' = \theta - \theta = 0, \quad R\,R' = R^2. \quad (36)$$

18. *Corollary.* The reverse of § 11 is, that *the modulus of a quotient is equal to the quotient of the modulus of the dividend divided by that of the divisor, and the argument of the quotient is equal to the argument of the dividend diminished by that of the divisor.*

Thus

$$\frac{R'(\cos.\theta'+\sin.\theta'.\sqrt{-1})}{R(\cos.\theta+\sin.\theta.\sqrt{-1})}$$
$$=\frac{R'}{R}[\cos.(\theta'-\theta)+\sin.(\theta'-\theta).\sqrt{-1}]. \quad (37)$$

19. *Corollary.* When $\theta'=0$, and $R'=1$; (38)

(37) becomes

$$\frac{1}{R(\cos.\theta+\sin.\theta.\sqrt{-1}}=\frac{1}{R}[\cos.(-\theta)+\sin.(-\theta).\sqrt{-1}],$$

or

$$[R(\cos.\theta+\sin.\theta.\sqrt{-1})]^{-1}=R^{-1}[\cos.(-\theta)+\sin.(-\theta).\sqrt{-1}]$$
$$=R^{-1}(\cos.\theta-\sin.\theta.\sqrt{-1}); \quad (39)$$

and raising to the nth power, by means of (30),

$$[R(\cos.\theta+\sin.\theta.\sqrt{-1})]^{-n}=R^{-n}[\cos.(-n\theta)+\sin.(-n\theta).\sqrt{-1}]$$
$$=R^{-n}(\cos.n\theta-\sin.n\theta\sqrt{-1}); \quad (40)$$

that is, *the rule of* § 12 *may be extended to the case of negative powers.*

20. *Corollary. The rule of* § 12 *may, then, be extended, by* § 1, *to all powers, real or imaginary.*

21. *Problem. To find the modulus and argument of the sum or difference of several imaginary qnantities.*

Solution. Let the given sum or difference be

$$r(\cos.\theta+\sin.\theta.\sqrt{-1})\pm r'(\cos.\theta'+\sin.\theta'.\sqrt{-1})\pm \&c., \quad (41)$$

and let R be its modulus, and Θ its argument; we have by (4 and 5) and by (9 and 29 of Trig.)

Imaginary sum or difference.

$$R^2 = (r\cos.\theta \pm r'\cos.\theta' \pm \&c.)^2 + (r\sin.\theta \pm r'\sin.\theta' \pm \&c.)^2$$
$$= r^2 + r'^2 + \&c. \pm 2rr'\cos.(\theta - \theta') \pm \&c., \qquad (42)$$
$$\tan.\Theta = \frac{r\sin.\theta \pm r'\sin.\theta' \pm \&c.}{r\cos.\theta \pm r'\cos.\theta' \pm \&c.} \qquad (43)$$

22. *Corollary.* Since every cosine is less than unity, (42) gives
$$R^2 < r^2 + r'^2 + \&c. \;\; + 2rr' + \&c.,$$
or
$$R^2 < (r + r' + \&c.)^2,$$
or
$$R < r + r' + \&c.; \qquad (44)$$
that is, the modulus of the sum or difference of several imaginary quantities is less than the sum of their moduli.

23. *Corollary.* When there are only two terms in (41), (42) becomes
$$R^2 = r^2 + r'^2 \pm 2rr'\cos.(\theta - \theta'); \qquad (45)$$
and, therefore,
$$R^2 > r^2 + r'^2 - 2rr',$$
or
$$R > r - r'; \qquad (46)$$
that is, the modulus of the sum or difference of two imaginary quantities is greater than the difference of their moduli.

CHAPTER II.

IMAGINARY INFINITESIMALS.

24. An *imaginary infinitesimal* is an imaginary quantity, whose modulus is an infinitesimal.

The order of an imaginary infinitesimal is the same with that of its modulus.

25. *Corollary.* It follows from Chapter II. of the Differential Calculus, and the preceding Chapter, that *all the propositions, which have hitherto been investigated respecting real infinitesimals, may be extended to imaginary infinitesimals.*

CHAPTER III.

IMAGINARY ROOTS OF EQUATIONS.

26. *Problem. To solve a binomial equation, and reduce all its imaginary roots to the form of* § 1.

Solution. Let the equation be

$$A\,x^a = M, \tag{47}$$

in which A and M are real or imaginary, and a a positive integer. When (47) is divided by A by means of § 18, it is reduced to the form

$$x^a = m, \tag{48}$$

in which m is of the form of § 1. Let then

$$m = r\,(\cos.\,\theta + \sin.\,\theta.\,\sqrt{-1}), \tag{49}$$

or

$$x^a = r\,(\cos.\,\theta + \sin.\,\theta.\,\sqrt{-1}). \tag{50}$$

The ath root of (50) is, by § 16,

$$x = \sqrt[a]{r}.\,(\cos.\,\frac{\theta}{a} + \sin.\,\frac{\theta}{a}.\sqrt{-1}). \tag{51}$$

27. *Scholium.* Since θ has, by (9), an infinity of values, (51) would at first sight appear to have a like infinity of values. But, by (9),

$$\frac{\theta}{a} = \frac{\theta_0}{a} \pm \frac{2\,n\,\pi}{a}, \tag{52}$$

whence the values of x are identical, when they correspond to values of θ, for which the difference of the values of n is equal to a, or is some multiple of a. Now, by subtracting from any value of n the greatest multiple of a contained in it, a remainder is obtained, which is less than a. The number of different values of x is, therefore, the same with the number of positive integers (zero included) which are less than a; that is, *the number of values of x or the number of roots of equation* (48) *is just equal to a.*

28. *Corollary.* When m is real and positive, (11) gives

$$x = \sqrt[a]{m}\left(\cos.\frac{2\,n\,\pi}{a} \pm \sin.\frac{2\,n\,\pi}{a}.\sqrt{-1}\right), \tag{53}$$

in which the double sign renders it unnecessary to notice those values of n which exceed the half of a.

29. *Corollary.* The value of n

$$n = 0, \tag{54}$$

reduces (53) to its real positive root

$$x = \sqrt[a]{m}. \tag{55}$$

30. *Corollary.* When a is even in (53), the value of n

$$n = \tfrac{1}{2}\,a, \tag{56}$$

gives

$$\frac{2\,n\,\pi}{a} = \pi, \tag{57}$$

$$x = -\sqrt[a]{m}. \tag{58}$$

31. *Corollary.* When m is real and negative, (12) gives

$$x = \sqrt[a]{-m}\left(\cos.\frac{2n+1}{a}\pi \pm \sin.\frac{2n+1}{a}\pi.\sqrt{-1}\right), \quad (59)$$

in which the double sign renders it unnecessary to notice those values of n which exceed the half of a.

32. *Corollary.* When a is odd, the value of n

$$n = \frac{a-1}{2}, \quad (60)$$

gives

$$\frac{2n+1}{a}\pi = \pi, \quad (61)$$

$$x = -\sqrt[a]{-m}. \quad (62)$$

33. *Theorem. Every equation has at least one real root or one imaginary root of the form* (1).

Proof. Let all the terms of the equation be transposed to its first member, which reduces it to the form

$$f.x = 0. \quad (63)$$

Let now x_0 be any real or imaginary value of x, for which the value of this first member neither vanishes, nor is infinite, and let h be an infinitesimal; let also $d^n_c.f.x_0$ be the first differential coefficient of $f.x_0$ which does not vanish; and (533 of Vol. I.) gives

$$f.(x_0+h) = f.x_0 + \frac{h^n}{1.2.3....n}\, d^n_c.f.x_0. \quad (64)$$

Again, let i be an assumed real infinitesimal, and let h be determined to satisfy the assumed binomial equation

$$\frac{h^n}{1.2.3....n} d_c^n . f.x_0 = - i f.x_0. \tag{65}$$

This value of h, being substituted in (64), gives

$$f.(x_0 + h) = f.x_0 - i f.x_0 = (1 - i) f.x_0; \tag{66}$$

so that if r is the modulus of $f.x_0$, that of $f.(x_0 + h)$ is, by § 11, $(1 - i) r$, and therefore less than that of $f.x_0$. The least possible modulus of $f.x$ is then less than r, unless r is zero; this least modulus must then be zero, and the corresponding value of x is a root of the equation (63).

34. *Scholium.* The preceding argument does not exclude infinity from being the root of the given equation, so that the following is a convenient statement of the above theorem;

Every equation has at least one finite root of the form (1), *when, after it is reduced to the form* (63), *it does not vanish for an infinite value of the variable.*

35. *Corollary.* If the first member of (63) is a polynomial of the form

$$x_n + a x^{n-1} + b x^{n-2} + \&c., \tag{67}$$

and if x' is a root of the equation, this polynomial must be divisible by $x - x'$; and the quotient must be a polynomial of the $(n - 1)$st degree, which must be divisible by a similar factor $x - x''$, and so on.

Hence (67) *must be the continued product of n different factors of the form* $(x - x')$; *that is, the equation*

$$x^n + a x^{n-1} + b x^{n-2} + \&c. = 0 \qquad (68)$$

must have n roots of the form (1), *whether a, b, &c. be real or imaginary.*

36. A *real function* is one, which has real values for all real values of the variable, and has not imaginary values, unless the variable is imaginary.

37. *Theorem.* The conjugate of a real function is the same function of the conjugate of the variable; or, algebraically, if

$$P + Q\sqrt{-1} = f.(p + q\sqrt{-1}), \qquad (69)$$

where $f.$ denotes a real function, then

$$P - Q\sqrt{-1} = f.(p - q\sqrt{-1}). \qquad (70)$$

Proof. The function, which is the second member of (69), may be developed and arranged according to powers of $\sqrt{-1}$. Let, then, the aggregate of all the terms which are independent of $\sqrt{-1}$, and of those which are multiplied by even powers of $\sqrt{-1}$ be denoted by P; while the aggregate of all those terms which are multiplied by odd powers of $\sqrt{-1}$, is denoted by Q'. The value of P is real, and remains unchanged by changing $\sqrt{-1}$ to $-\sqrt{-1}$, while that of Q' is reversed; that is, the value of the function is changed

$$\text{from} \quad P + Q' \quad \text{to} \quad P - Q'. \qquad (71)$$

But the quotient of Q' divided by $\sqrt{-1}$, containing only even powers of $\sqrt{-1}$, is a real quantity, which may be denoted by Q, that is,

$$Q' = Q\sqrt{-1}, \tag{72}$$

$$P + Q' = P + Q\sqrt{-1}; \tag{73}$$

so that by reversing the sign of $\sqrt{-1}$, (69) is changed to (70).

38. *Corollary*. When $$Q = 0, \tag{74}$$
(69 and 70) become

$$P = f\cdot(p + q\sqrt{-1}) = f\cdot(p - q\sqrt{-1}); \tag{75}$$

that is, every real value of a real function corresponds to two different values of the variable, which are conjugate to each other.

39. *Corollary*. When $$P = 0, \tag{76}$$
(75) becomes

$$0 = f\cdot(p + q\sqrt{-1}) = f\cdot(p - q\sqrt{1}); \tag{77}$$

that is, *when the function, which is the first member of* (63), *is real, the conjugate of every imaginary root is also a root of the equation.*

40. *Corollary*. If x' is a root of the equation (68), when a, b, &c. are real, and if x'' is the conjugate of x', x'' is also a root of this equation, and the first member is divisible by the product

$$(x - x')(x - x'') = x^2 - (x' + x'')x + x'x''. \tag{78}$$

Number of real factors of a real polynomial.

If r is the modulus of x' and θ its argument, (8, 24, and 36) give

$$x' + x'' = 2\, r \cos.\, \theta, \quad x'\, x'' = r^2; \tag{79}$$

whence (78) becomes the real factor

$$x^2 - 2\, r\, x \cos.\, \theta + r^2; \tag{80}$$

so that *any real polynomial of the form* (67) *is the continued product of as many real factors of the form* $x - x'$ *as the equation* (68) *has real roots, multiplied by the continued product of half as many real quadratic factors of the form* (80) *as* (68) *has imaginary roots.*

41. Examples.

1. Decompose $x^7 - b^7$ into a continued product of real factors of the first and second degree.

Solution. The equation

$$x^7 - b^7 = 0, \quad \text{or} \quad x^7 = b^7,$$

gives in (48) $\qquad m = b^7, \quad a = 7;$

whence (53) becomes

$$x = b\,(\cos.\, \tfrac{2}{7}\, n\, \pi \pm \sin.\, \tfrac{2}{7}\, n\, \pi \,.\, \sqrt{-1});$$

which becomes, by putting successively for n all integers less than half of 7,

$$x = b,$$

$$x = b\,(\cos.\, \tfrac{2}{7}\, \pi \pm \sin.\, \tfrac{2}{7}\, \pi \,.\, \sqrt{-1}),$$

$$x = b\,(\cos.\, \tfrac{4}{7}\, \pi \pm \sin.\, \tfrac{4}{7}\, \pi \,.\, \sqrt{-1}),$$

$$x = b\,(\cos.\, \tfrac{6}{7}\, \pi \pm \sin.\, \tfrac{6}{7}\, \pi \,.\, \sqrt{-1});$$

Decomposition of a function into real factors.

so that, by (80), the continued product is

$$x^7 - b^7 = (x - b)\,(x^2 - 2\,b\,x\ \text{cos.}\ \tfrac{2}{7}\,\pi + b^2)$$
$$(x^2 - 2\,b\,x\ \text{cos.}\ \tfrac{4}{7}\,\pi + b^2)\,(x^2 - 2\,b\,x\ \text{cos.}\ \tfrac{6}{7}\,\pi + b^2).$$

2. Decompose $x^4 + b^4$ into a product of real factors of the first and second degree.

Solution. The equation

$$x^4 + b^4 = 0, \quad \text{or} \quad x^4 = -\,b^4,$$

gives in (48)

$$m = -\,b^4, \quad -\,m = b^4, \quad a = 4;$$

whence (59) becomes

$$x = b\,(\text{cos.}\ \tfrac{1}{4}\,(2\,n + 1)\,\pi \pm \text{sin.}\ \tfrac{1}{4}\,(2\,n + 1)\,\pi\,.\,\sqrt{-1});$$

which becomes, by putting successively for n all integers less than 2,

$$x = b\,(\text{cos.}\ \tfrac{1}{4}\,\pi \pm \text{sin.}\ \tfrac{1}{4}\,\pi\,.\,\sqrt{-1}) = b\,(\tfrac{1}{2}\sqrt{2} \pm \tfrac{1}{2}\sqrt{2}\,.\,\sqrt{-1}),$$
$$x = b\,(\text{cos.}\ \tfrac{3}{4}\,\pi \pm \text{sin.}\ \tfrac{3}{4}\,\pi\,.\,\sqrt{-1}) = b\,(-\tfrac{1}{2}\sqrt{2} \mp \tfrac{1}{2}\sqrt{2}\,.\,\sqrt{-1});$$

so that, by (80), the continued product is

$$x^4 + b^4 = (x^2 - 2\,b\,x\,\text{cos.}\ \tfrac{1}{4}\,\pi + b^2)\,(x^2 - 2\,b\,x\,\text{cos.}\ \tfrac{3}{4}\,\pi + b^2)$$
$$= (x^2 - \sqrt{2}\,.\,b\,x + b^2)\,(x^2 + \sqrt{2}\,.\,b\,x + b^2).$$

3. Decompose $x^4 - b^4$ into a continued product of real factors of the first and second degree.

Ans. $(x - b)\,(x + b)\,(x^2 + b^2)$.

4. Decompose $x^5 + b^5$ into a continued product of real factors of the first and second degree.

Ans. $(x + b)\,(x^2 - 2\,b\,x\,\text{cos.}\,\tfrac{1}{5}\,\pi + b^2)\,(x^2 - 2\,b\,x\,\text{cos.}\ \tfrac{3}{5}\,\pi + b^2)$.

Decomposition into real factors.

5. Decompose $x^6 - b^6$ into a continued product of real factors of the first and second degree.

Ans. $(x - b)(x + b)(x^2 + bx + b^2)(x^2 - bx + b^2)$.

6. Decompose $x^6 + b^6$ into a continued product of real factors of the first and second degree.

Ans. $(x^2 + \sqrt{3}.bx + b^2)(x^2 + b^2)(x^2 - \sqrt{3}.bx + b^2)$.

CHAPTER IV.

IMAGINARY EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

42. *Problem. To reduce an imaginary power of a real quantity to the form* (1).

Solution. Let the exponent of the power be $A + B\sqrt{-1}$, and let R be the modulus and Θ the argument of this power of the real quantity a, that is, let

$$a^{A+B\sqrt{-1}} = R\,(\cos.\,\Theta + \sin.\,\Theta\,.\,\sqrt{-1}). \qquad (81)$$

The infinitesimal power i of this equation is by (28)

$$a^{i(A+B\sqrt{-1})} = R^i\,(\cos.\,i\,\Theta + \sin.\,i\,\Theta\,.\,\sqrt{-1}). \qquad (82)$$

Hence by (418 of Vol. I. and § 22 of Plane Trig.)

$$\begin{aligned} 1 + i\,(A + B\sqrt{-1})\log.\,a &= (1 + i\log.\,R)\,(1 + i\,\Theta\sqrt{-1}) \\ &= 1 + i\,(\log.\,R + \Theta\sqrt{-1}). \qquad (83) \end{aligned}$$

Hence, by § 8, and using

$$e = \text{the base of the Neperian logarithms}, \qquad (84)$$

$$\log.\,R = A\log.\,a = \log.\,a^A, \qquad R = a^A, \qquad (85)$$

$$B\log.\,a = \Theta = \log.\,a^B, \qquad a^B = e^\Theta; \qquad (86)$$

which, substituted in (81), give

$$a^{A+B\sqrt{-1}} = a^A\,(\cos.\,B\log.\,a + \sin.\,B\log.\,a\,.\,\sqrt{-1}). \qquad (87)$$

Imaginary logarithm.

43. *Corollary.* When $A = 0,$
(87) becomes

$$a^{B\sqrt{-1}} = \cos. B \log. a + \sin. B \log. a . \sqrt{-1}. \qquad (88)$$

44. *Corollary.* When $a = e,$
(87 and 88) become

$$e^{A+B\sqrt{-1}} = e^{A}(\cos. B + \sin. B . \sqrt{-1}), \qquad (89)$$

$$e^{B\sqrt{-1}} = \cos. B + \sin. B . \sqrt{-1}. \qquad (90)$$

45. *Corollary.* Reversing the sign of B, (89 and 90) become

$$e^{A-B\sqrt{-1}} = e^{A}(\cos. B - \sin. B . \sqrt{-1}), \qquad (91)$$

$$e^{-B\sqrt{-1}} = \cos. B. - \sin. B . \sqrt{-1}. \qquad (92)$$

46. *Problem. To reduce the logarithm of an imaginary quantity to the form* (1).

Solution. Let r be the modulus and θ the argument of the imaginary quantity, and (90) gives

$$r(\cos. \theta + \sin. \theta . \sqrt{-1}) = r\, e^{\theta\sqrt{-1}}; \qquad (93)$$

the logarithm of which is

$$\begin{aligned}\log. [r(\cos. \theta + \sin. \theta . \sqrt{-1})] &= \log. r + \log. e^{\theta\sqrt{-1}} \\ &= \log. r + \theta\sqrt{-1}. \qquad (94)\end{aligned}$$

47. *Corollary.* By (4, 5, and 94)

$$\begin{aligned}\log.(A+B\sqrt{-1}) &= \log.\sqrt{(A^2+B^2)} + \tan.^{[-1]}\frac{B}{A}.\sqrt{-1} \\ &= \tfrac{1}{2}\log. (A^2+B^2) + \tan.^{[-1]}\frac{B}{A}.\sqrt{-1}; \qquad (95)\end{aligned}$$

and as there is an infinity of values of

$$\theta = \tan.^{[-1]} \frac{B}{A},$$

every quantity, real or imaginary, has an infinity of logarithms, of which there is never more than one real logarithm, and that, by § 5, only when the quantity is real and positive.

48. *Corollary.* By § 5, when A is positive, and

$$B = 0,$$

(95) becomes

$$\log. A = \log. A \pm 2\,n\,\pi\,\sqrt{-1}, \tag{96}$$

in which log. A of the second member is the real value of this logarithm.

49. *Corollary.* By § 5, when A is negative and

$$B = 0,$$

(95) becomes

$$\log. A = \log. (-A) \pm (2\,n + 1)\,\pi\,\sqrt{-1}. \tag{97}$$

50. Examples.

1. What is the logarithm of $\frac{1}{2}\sqrt{2}\,(1 + \sqrt{-1})$?

Ans. $(\frac{1}{4} \pm 2\,n)\,\pi\,\sqrt{-1}$.

2. What is the logarithm of $\sqrt{3} + \sqrt{-1}$?

Ans. $\log. 2 + (\frac{1}{6} \pm 2\,n)\,\pi\,\sqrt{-1}$.

CHAPTER V.

IMAGINARY CIRCULAR FUNCTIONS.

51. *Problem. To reduce the sine and cosine of an imaginary angle to the form* (1).

Solution. *a.* Let the angle be $B\sqrt{-1}$, which being substituted for B in (90 and 92), gives

$$e^{-B} = \cos. B\sqrt{-1} + \sin. B\sqrt{-1} \,.\, \sqrt{-1}, \qquad (98)$$

$$e^{B} = \cos. B\sqrt{-1} - \sin. B\sqrt{-1} \,.\, \sqrt{-1}. \qquad (99)$$

One half of the sum of (98 and 99) is

$$\cos. B\sqrt{-1} = \tfrac{1}{2}\,(e^{B} + e^{-B}). \qquad (100)$$

One half of the difference of (98 and 99), multiplied by $\frac{1}{2}\sqrt{-1}$, is

$$\sin. B\sqrt{-1} = \tfrac{1}{2}\,(e^{B} - e^{-B})\sqrt{-1}. \qquad (101)$$

b. When the angle is $A + B\sqrt{-1}$, (100 and 101) give

$$\sin.(A+B\sqrt{-1}) = \sin. A\cos. B\sqrt{-1} + \cos. A\sin. B\sqrt{-1}$$
$$= \tfrac{1}{2}\sin. A\,(e^{B}+e^{-B}) + \tfrac{1}{2}\cos. A\,(e^{B}-e^{-B})\sqrt{-1}; \qquad (102)$$

$$\cos.(A+B\sqrt{-1}) = \cos. A\cos. B\sqrt{-1} - \sin. A\sin. B\sqrt{-1}$$
$$= \tfrac{1}{2}\cos. A\,(e^{B}+e^{-B}) - \tfrac{1}{2}\sin. A\,(e^{B}-e^{-B})\sqrt{-1}. \qquad (103)$$

The imaginary angle, whose sine exceeds unity.

52. ***Problem.*** To reduce the imaginary angle, the absolute value of whose sine is greater than unity, to the form (1).

Solution. Let the given sine of the angle be $\pm(1+a)$, and let the required angle be $A+B\surd-1$; it is evident from (102) that, when the sine of the angle is real,

$$\cos. A\,(e^B - e^{-B}) = 0; \qquad (104)$$

that is, either

$$e^B = e^{-B}, \qquad (105)$$

whence

$$e^{2B} = 1, \quad 2B = 0, \quad B = 0; \qquad (106)$$

in which case the given angle is real, and the absolute value of its sine cannot exceed unity;

or

$$\cos. A = 0, \quad A = n\pi, \qquad (107)$$

$$\sin. A = \pm 1, \qquad (108)$$

whence, by (102 and 103), (109)

$$\sin.(A+B\surd-1) = \sin.(n\pi+B\surd-1)$$
$$= \pm\tfrac{1}{2}(e^B+e^{-B}) = \pm(1+a), \qquad (110)$$

$$\cos.(A+B\surd-1) = \cos.(n\pi+B\surd-1)$$
$$= \mp\tfrac{1}{2}(e^B-e^{-B})\surd-1$$
$$= \mp\surd(-2a-a^2) = \mp\surd(2a+a^2).\surd-1. \qquad (111)$$

The sum of (110), and (111) multiplied by $\surd-1$, is

$$e^B = 1+a\pm\surd(2a+a^2), \qquad (112)$$

whence

$$B = \log.[1+a\pm\surd(2a+a^2)]$$
$$= \pm\log.[1+a+\surd(2a+a^2)], \qquad (113)$$

and the angle is

$$n\pi \pm \log.[1+a+\surd(2a+a^2)].\surd-1. \qquad (114)$$

53. Examples.

1. Reduce tang. $(A + B\sqrt{-1})$ to the form (1).

$$Ans. \quad \frac{2 \sin. 2A}{e^{2B} + e^{-2B} + 2\cos. 2A} + \frac{(e^{2B} - e^{-2B})\sqrt{-1}}{e^{2B} + e^{-2B} + 2\cos. 2A}. \quad (115)$$

2. Reduce tang. $B\sqrt{-1}$ to the form (1).

$$Ans. \quad \frac{(e^{B} - e^{-B})\sqrt{-1}}{e^{B} + e^{-B}} = \frac{(e^{2B} - 1)\sqrt{-1}}{e^{2B} + 1}. \quad (116)$$

3. Reduce tang.$^{[-1]}$ $B\sqrt{-1}$ to the form (1).

Ans. When B is absolutely less than unity, it is

$$\pm n\pi + \tfrac{1}{2}\,[\log.(1 + B) - \log.(1 - B)]\,.\sqrt{-1}. \quad (117)$$

When B is positive and greater than 1, it is

$$\pm (n + \tfrac{1}{2})\,\pi + \tfrac{1}{2}\,[\log.(B+1) - \log.(B-1)]\,.\sqrt{-1}. \quad (118)$$

When B is negative and less than -1, it is

$$\pm (n + \tfrac{1}{2})\pi + \tfrac{1}{2}[\log. -(1+B) - \log.(1-B)].\sqrt{-1}. \quad (119)$$

When $B = \pm 1$, it is

$$A \pm \infty\,.\sqrt{-1}. \quad (120)$$

54. Equations (100 and 101) have suggested a new form of notation of great practical value, and for which tables have been constructed, similar to the common trigonometric tables. It consists in representing $-\sqrt{-1}\,.\,\sin. B\sqrt{-1}$ and $\cos. B\sqrt{-1}$ by Sin. B and Cos. B, which only differ in their initial capital letters from the common trigonometric no-

tation; this notation may also be extended to the other trigonometric functions. These new functions are called *potential functions.* We have, then,

$$\text{Sin.}\, B = -\sqrt{-1}.\,\sin.\, B\sqrt{-1} = \frac{\sin.\, B\sqrt{-1}}{\sqrt{-1}}$$

$$= \tfrac{1}{2}\,(e^{B} - e^{-B}), \qquad (121)$$

$$\text{Cos.}\, B = \cos.\, B\sqrt{-1} \;\; = \tfrac{1}{2}\,(e^{B} + e^{-B}), \qquad (122)$$

$$\text{Tang.}\, B = \frac{\text{Sin.}\, B}{\text{Cos.}\, B} = \frac{\text{tang.}\, B\sqrt{-1}}{\sqrt{-1}}, \text{ \&c.} \qquad (123)$$

55. *Corollary.* The differentiation of (121 - 123) gives

$$d_{c}.\,\text{Sin.}\, B = \tfrac{1}{2}\,(e^{B} + e^{-B}) = \text{Cos.}\, B, \qquad (124)$$

$$d_{c}.\,\text{Cos.}\, B = \tfrac{1}{2}\,(e^{B} - e^{-B}) = \text{Sin.}\, B, \qquad (125)$$

$$d_{c}\text{Tan.}\, B = \frac{1}{\cos.^{2} B\sqrt{-1}} = \frac{1}{\text{Cos.}^{2} B} = \text{Sec.}^{2} B. \quad (126)$$

56. EXAMPLES.

Demonstrate the following equations.

1. $$\text{Cos.}^{2}\, B - \text{Sin.}^{2}\, B = 1. \qquad (127)$$

Solution. By (121 and 122)

$$\text{Cos.}^{2}\, B = \tfrac{1}{4}\,(e^{2B} + 2 + e^{-2B})$$

$$\text{Sin.}^{2}\, B = \tfrac{1}{4}\,(e^{2B} - 2 + e^{-2B})$$

Hence $\quad \text{Cos.}^{2}\, B - \text{Sin.}^{2}\, B = 1.$

2. $\text{Sin.}\,(B \pm B') = \text{Sin.}\, B\, \text{Cos.}\, B' \pm \text{Cos.}\, B\, \text{Sin.}\, B' \qquad (128)$

3. $\text{Cos.}\,(B \pm B') = \text{Cos.}\, B\, \text{Cos.}\, B' \pm \text{Sin.}\, B\, \text{Sin.}\, B' \qquad (129)$

4. $\text{Sin.}\,(B + B') + \text{Sin.}\,(B - B') = 2\,\text{Sin.}\, B\, \text{Cos.}\, B' \qquad (130)$

Potential functions.

5. $\text{Sin.}(B+B') - \text{Sin.}(B-B') = 2\,\text{Cos.}B\,\text{Sin.}B'$ (131)

6. $\text{Cos.}(B+B') + \text{Cos.}(B-B') = 2\,\text{Cos.}B\,\text{Cos.}B'$ (132)

7. $\text{Cos.}(B+B') - \text{Cos.}(B-B') = 2\,\text{Sin.}B\,\text{Sin.}B'$ (133)

8. $\dfrac{\text{Sin.}\,B + \text{Sin.}\,B'}{\text{Sin.}\,B - \text{Sin.}\,B'} = \dfrac{\text{Tang.}\,\frac{1}{2}\,(B+B')}{\text{Tang.}\,\frac{1}{2}\,(B-B')}$ (134)

9. $\dfrac{\text{Cos.}\,B - \text{Cos.}\,B'}{\text{Cos.}\,B + \text{Cos.}\,B'} = \text{Tan.}\tfrac{1}{2}(B+B')\,\text{Tan.}\tfrac{1}{2}(B-B')$ (135)

10. $\text{Sin.}\,2\,B = 2\,\text{Sin.}\,B\,\text{Cos.}\,B$ (136)

11. $\text{Cos.}\,2\,B = \text{Cos.}^2\,B + \text{Sin.}^2\,B$ (137)

$$= 1 + 2\,\text{Sin.}^2\,B$$

$$= 2\,\text{Cos.}^2\,B - 1$$

12. $\text{Sin.}\,\frac{1}{2}\,B = \surd[\frac{1}{2}\,(\text{Cos.}\,2\,B - 1)]$ (138)

13. $\text{Cos.}\,\frac{1}{2}\,B = \surd[\frac{1}{2}\,(\text{Cos.}\,2\,B + 1)]$ (139)

14. $\text{Tang.}\,\frac{1}{2}\,B = \surd\left(\dfrac{\text{Cos.}\,2\,B - 1}{\text{Cos.}\,2\,B + 1}\right)$ (140)

15. $\text{Tang.}(B \pm B') = \dfrac{\text{Tang.}\,B \pm \text{Tang.}\,B'}{1 \pm \text{Tang}\;\;B\,\text{Tang.}\,B'}$ (141)

16. $\text{Tang.}\,2\,B = \dfrac{2\,\text{Tang.}\,B}{1 + \text{Tang.}^2\,B}.$ (142)

17. $d_c.\,\text{Sin.}^{[-1]}\,x = (1+x^2)^{-\frac{1}{2}} = \dfrac{1}{\surd(1+x^2)}.$ (143)

Solution. Let $x = \text{Sin.}\,B$, or $B = \text{Sin.}^{[-1]}\,x$

Then by (124 and 127)

$$d_{cB}.\,x = \text{Cos.}\,B = \surd(1 + \text{Sin.}^2 B) = \surd(1 + x^2)$$

3*

Potential functions.

and by (Vol. I. 566)

$$d_c \text{Sin.}^{[-1]} x = d_{c.x} B = \frac{d_{c.B} . B}{d_{c.B} . x} = \frac{1}{\sqrt{(1+x^2)}}$$

18. $$d_c . \text{Cos.}^{[-1]} x = (x^2 - 1)^{-\frac{1}{2}} = \frac{1}{\sqrt{(x^2-1)}} \qquad (144)$$

19. $$d_c . \text{Tang.}^{[-1]} x = \frac{1}{1-x^2}. \qquad (145)$$

20. $$\text{Sin.}\, x = x + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c. \qquad (146)$$

21. $$\text{Cos.}\, x = 1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \&c. \qquad (147)$$

CHAPTER VI.

REAL ROOTS OF NUMERICAL EQUATIONS.

57. While the imaginary roots of equations are of great subsidiary value in mathematical investigations, and frequently admit of curious and interesting interpretations in physical inquiries, real roots are the primary objects of attention, and methods of determining their numerical values are exceedingly important in practice. *Stern's method* is the simplest which has yet been published, and is of almost universal application.

58. If the values of a given function and of its successive differential coefficients, as far as the nth, are found for a given value a of the variable; and if the successive signs of these values are placed after each other, the row of signs thus formed is, in this chapter, called *the nth row of signs* (a), or simply the *nth row* (a), or *the row* (a); any pair of successive signs in this row is called a *permanence*, when the signs are alike, and a *variation*, when the signs are unlike.

59. *Theorem.* If a function and its differential coefficients inferior to the nth all vanish, but the nth does not vanish, for a value a of the variable, the nth row of signs $(a + i)$, i being an infinitesimal, consists

wholly of permanences, while the nth row $(a - i)$ consists wholly of variations.

Proof. It follows from (Vol. I. 533), that if $f.\,x$ is the given function

$$f.\,(a+i) = \frac{i^n}{1.2.3\ldots n}.\,d_c^n.\,f.\,a; \qquad (148)$$

the differential coefficients of which, taken relatively to i, are

$$d_c.\,f.\,(a+i) = \frac{i^{n-1}}{1.2.3\ldots(n-1)}.\,d_c^n.\,f.\,a = \frac{n\,f.\,(a+i)}{i},$$

$$d_c^2.\,f.\,(a+i) = \frac{i^{n-2}}{1.2.3\ldots(n-2)}.\,d_c^n.\,f.\,a = \frac{(n-1)\,f.\,(a+i)}{i},$$

&c. &c. &c. (149)

that is, all the terms of the series

$$f.\,(a+i),\quad d_c.\,f.\,(a+i),\quad d_c^2.\,f.\,(a+i),\ \&c. \qquad (150)$$

have the same sign.

But the reversing of the sign of i in these equations gives

$$d_c.\,f.\,(a-i) = -\frac{n\,f.\,(a-i)}{i},$$

$$d_c^2.\,f.\,(a-i) = -\frac{(n-1)\,f.\,(a-i)}{i},\ \&c. \qquad (151)$$

that is, the signs of any two successive terms in the series

$$f.\,(a-i),\quad d_c.\,f.\,(a-i),\quad d_c^2.\,f.\,(a-i),\ \&c. \qquad (152)$$

are unlike, and the terms are alternately positive and negative.

60. *Corollary.* If, in a series of the successive differential coefficients of a function terminating with the

nth, all vanish except the nth for a value a of the variable, the signs of this series will in the row of signs $(a+i)$ constitute a series of permanences, and in the row $(a-i)$, a series of variations.

61. *Theorem. If the first member of the equation*

$$f.'x = 0 \tag{153}$$

is continuous between the values a and b of the variable, a being greater than b, if the number of permanences in the nth row of signs (a) exceeds the number of permanences in the nth row (b), and if the excess is denoted by v, the number of real roots of (153), *which are included between a and b, cannot exceed v.*

Proof. For while the value of x varies from a to b, a change of sign can occur in the row of signs, only when $f.\,x$, or one of its differential coefficients, or a series of them, passes through zero. Now, the case of a single function being included in that of a series, when a series of these functions vanishes, a number of permanences must, by §§ 59 and 60, be lost, equal to the number of functions. If, then, this series begins with $f.\,x$, as it must when the variable is equal to a root of the equation, one permanence, at least, must be lost; that is, *there is a loss of one or more permanences in the row of signs, corresponding to every real root of the equation.*

If the vanishing series does not begin with $f.\,x$, and consists of an even number of functions, the sign of its first function is, by (148–152), the same with that of the function which follows the series, both before and after vanishing. The relation of the first sign of the series to the sign which pre-

cedes the series is, therefore, unchanged; and *the loss of permanences is exactly equal to the even number of terms of the vanishing series.*

If the vanishing series consists of an odd number of functions, the sign of its first function is reversed when it vanishes. If, therefore, it has, before it vanishes, the same sign with the preceding function, another permanence is here lost, which is to be added to those before noticed. But if it has, before it vanishes, the opposite sign to the preceding function, a new permanence is introduced, when it vanishes, which is to be subtracted from the number of the others. *In one case, therefore, the whole number of lost permanences is one greater than the odd number of terms in the vanishing series; and, in the other case, it is one less than this number.*

In any case, the number of lost permanences is, at least, as great as the number of real roots of the equation.

62. *Corollary.* When the loss of permanences does not arise from a real root of the equation, the number of lost permanences is even; *so that if the number of lost permanences is odd, that is, if v is odd, the equation must have at least one real root between a and b.*

63. *Problem. To find all the real roots of an equation.*

Solution. Reduce the equation to the form (153), *simplify it as much as possible; and determine, as nearly as possible by inspection, those limits between which the different real roots must be, if there are any.*

Find the successive differential coefficients of the first

member, until one is obtained which does not vanish between two limits a and b, between which there may be real roots. Let this be the nth differential coefficient.

If, then, a being greater than b, the number of permanences in the nth row of signs (a) is the same with that in the row (b), there is no real root between a and b. If the difference between the number of permanences is even, the question of a real root between a and b is undecided; and if this difference is odd, there must be such a root.

Let, then, the mth differential coefficient be the highest one, of which the sign is different in the row (a) and in the row (b). The equation

$$d_c.^{m+1} f.\, x = 0, \qquad (154)$$

can then have no real root between a and b, while the equation

$$d_c.^{m} f.\, x = 0, \qquad (155)$$

must have one, which can be found by the process given in the sequel of this solution. If c is the root of (155), it may also be a root of (153), which can be discovered by trial.

However this may be, the preceding process is to be repeated for the limits a and c + i, i being an infinitesimal, and also for the limits c + i, and b, using the mth row of signs instead of the nth. A continuation of the process must finally lead to a division of the limits from a to b, into sets of limits so narrow, that, be-

tween each set there can only be one real root of (153) *and no real root of the equation*

$$d_c \cdot f. x = 0. \tag{156}$$

Let a' and b' be a set of these limits, and if they are far apart, substitute for x, in the first member of (153), *different numbers, the various integers for instance, between a' and b', until one is found which does not differ much from the required root, and denote this first approximation to the root by x_0. Then, if the exact root is $x_0 + h$, we have by* (Vol. I. 532)

$$f. (x_0 + h) = f. x_0 + h\, d_c. f. (x_0 + \theta\, h) = 0, \tag{157}$$

whence, by neglecting $\theta\, h$, the approximate value of h is obtained, which is

$$h = -\frac{f. x_0}{d_c. f. x_0} \tag{158}$$

and from the new approximation to the root $x_0 + h$, which is thus found, a new approximation can be obtained; and so on, to any required degree of accuracy.

64. *Corollary.* The rate of approximation can be readily determined; for if two successive values of h are h and h' corresponding to x_0 and x_0', so that

$$x_0' = x_0 + h \tag{159}$$

the error of x_0 differs from h by a quantity much smaller than h; and that of x_0' is nearly equal to h'. Now suppose

$$h < (\tfrac{1}{10})^{\epsilon} \tag{160}$$

Rate of approximation.

and we have by (158) and by Taylor's theorem

$$f.x_0' = f(x_0+h) = f.x_0 + d_e.f.x_0.h + \tfrac{1}{2} d_e^2.f.x_0'.h^2 + \&c. \quad (161)$$

$$d_e.f.x_0' = d_e.f.x_0 + \&c. \quad (162)$$

but by (158)

$$f.x_0 + d_e.f.x_0.h = 0 \quad (163)$$

$$f.x_0' = \tfrac{1}{2} d_e^2.f.x_0.h^2 + \&c. \quad (164)$$

whence neglecting h^3, &c.

$$h' = -\frac{f.x_0'}{d_e.f.x_0} = -\frac{d_e^2.f.x_0}{2d_e.f.x_0} h^2 \quad (165)$$

If now we find

$$\frac{d_e^2.f.x_0}{2d_e.f.x^0} < (\tfrac{1}{10})^k \quad (166)$$

we have, neglecting the signs,

$$h' < (\tfrac{1}{10})^{2s+k} \quad (167)$$

and therefore if one approximation is accurate to s places of decimals, the next will be accurate to 2 s + k places.

65. *Corollary.* Since the real root is exactly

$$x = x_0 + h;$$

we have $$x_0 = x - h, \quad (168)$$

whence by (153 and Vol. I. 532)

$$f.x_0 = f(x-h) = f.x - h.d_e.f.(x - \theta h)$$
$$= -h\, d_e.f.(x - \theta h), \quad (169)$$

or neglecting θh

$$f.x_0 = -h\, d_e.f.x = (x_0 - x)\, d_e.f.x. \quad (170)$$

In the same way for another hypothesis x_0', we have

$$f.x_0' = -h'\, d_e.f.x = (x_0' - x)\, d_e.f.x. \quad (171)$$

4

The difference of (170) and (171) is

$$f.x_0 - f.x'_0 = (x_0 - x'_0)\, d_c.f.x \qquad (172)$$

and the quotient of (171) by (172), is

$$\frac{f.x'_0}{f.x_0 - f.x'_0} = \frac{x'_0 - x}{x_0 - x'_0} \qquad (173)$$

which is identical with the famous ***rule of false***, or ***rule of double position***, in arithmetic; and this admirable rule, the principle of which is obviously at the foundation of all higher mathematics, and pervades all practical science in some form or other, is sufficient for obtaining, with ease and accuracy, the most important numerical results.

66. Examples.

1. Solve the equation

$$x \log.' x - 100 = 0$$

in which log.' denotes the common tabular logarithms.

Solution. The theory of logarithms gives

$$\log.' x = \log.' e \,.\, \log.\, x.$$

Hence if

$$f.x = x \log.' x - 100$$

$$d_c.f.x = \log.' x + \log.' e$$

$$d_c^2.f.x = \frac{\log.' e}{x}.$$

The value of $d_c^2.f.x$ is positive between the limits

$$x = 0, \quad \text{and} \quad x = \infty$$

and $d_c.f.x$ is negative between the limits

$$x = 0, \quad \text{and} \quad x = e^{-1}$$

at both which limits $f.x$ is negative, and the given equation has therefore no real roots between these limits. But $d_c.f.x$ is positive between the limits $x = e^{-1}$ and $x = 0$, at which limits $f.x$ has opposite signs, and the given equation has, therefore, only one real root, which is between these limits.

A very few trials show, then, that the root is not far from 60, for which value

$$f.x = 7, \quad d_c^1.f.x = 1{\cdot}78 + 0{\cdot}43 = 2{\cdot}21$$

$$d_c^2.f.x = {\cdot}0072, \quad k = 2$$

and the rest of the calculation may be arranged as in the following form, in the first column of which are placed the successive values of $f.x$, in the second those of $d_c.f.x$, and in the third those of x.

		60
7.	2·21	57
0·084860	2·19017	56·9612

Ans. 56·9612.

2. Solve the equation

$$x - \cos.\, x = 0. \qquad \textit{Ans. } 0{\cdot}7391.$$

3. Solve the equation

$$x - \text{tang.}\, x = 0.$$

Ans. There are an infinity of roots, one being contained between each set of limits

$$n\,\pi \text{ and } (n + \tfrac{1}{2})\,\pi$$

in which any integer may be substituted for n, the value between π and $\frac{3}{2}\pi$ is 4·4934.

BOOK IV.

RESIDUAL CALCULUS.

BOOK IV.

RESIDUAL CALCULUS.

CHAPTER I.

RESIDUATION.

1. For every finite value of x, which satisfies the equation

$$f{\cdot}x = \infty\,, \text{ that is, } \frac{1}{f.x} = (f.x)^{-1} = 0, \qquad (174)$$

the first term of Taylor's theorem (Vol. I. 442) is infinite, and the development of $f.(x+h)$ by that theorem is impossible. In this case, if i is an infinitesimal, $f.(x+i)$ is infinite; and if we suppose it to be of the mth order of infinity, the expression

$$i^m f.(x+i) \qquad (175)$$

is of the order zero, and is usually finite, as in § 26 of the Differential Calculus. The quantity

$$h^m f.(x+h) \qquad (176)$$

may then be developed, by MacLaurin's Theorem (445, Vol. I.), as a function of h, and the result will be of the form

$$h^m f.(x+h) = A + B\,h + \&c.$$
$$+ Q\,h^{m-2} + Rh^{m-1} + S\,h^m + T\,h^{m+1} + \&c. \quad (177)$$

which, divided by h^m, gives

$$f.(x+h) = A\,h^{-m} + B\,h^{-(m-1)} + \&c.$$
$$+ Q\,h^{-2} + R\,h^{-1} + S + T\,h + \&c. \quad (178)$$

that is, $f.(x+h)$ *can, even for a value of* x *which satisfies* (174), *be developed in a series consisting of two parts, one of which*

$$S + T\,h + \&c. \quad (179)$$

is, like Taylor's Theorem, arranged according to positive and ascending powers of h, *and the other part*

$$R\,h^{-1} + Q\,h^{-2} + \&c. + B\,h^{-(m-1)} + A\,h^{-m} \quad (180)$$

is arranged according to negative and descending powers of h.

2. The coefficient of h^{-1}, in the development of $f.(x+h)$ by the preceding method, is called the *residual* of $f.x$, and vanishes for all values of x, except those which satisfy (174).

To *residuate* is to find the residual.

3. *Problem. To residuate a given function.*

Solution. Let $f.$ denote the given function, and let x_0 be the value of x which satisfies (174). Since R, which is the residual of this function by (180), is the coefficient of h^{m-1} in (177) the development of $h^m f.(x_0+h)$ by MacLaurin's Theorem; we have by (445 of Vol. I.), if we regard h as the variable,

$$R = \frac{d_{c.h}^{m-1}.\,h^m f.(x_0+h)}{1.2.3.\ldots(m-1)}, \quad (181)$$

provided that after the differentiation we put

$$h = 0.$$

This vanishing of h may be effected in the general form, by substituting for h the infinitesimal i, which gives

$$R = \frac{d_{c.i}^{m-1} \,.\, i^m f.(x_0 + i)}{1.2.3\ldots(m-1)}. \tag{182}$$

4. Examples.

1. To residuate the function $(x-a)^{-1}(x-b)^{-2}$.

Solution. This function becomes infinite of the first order, when

$$x = a + i;$$

and infinite of the second order, when

$$x = b + i.$$

The residual which corresponds to $x = a$, is, then,

$$i(i)^{-1}(a-b+i)^{-2} = (a-b)^{-2};$$

and that which corresponds to $x = b$, is

$$d_{c.i}.i^2(b-a+i)^{-1}i^{-2} = d_{c.i}.(b-a+i)^{-1} = -(b-a+i)^{-2}$$
$$= -(b-a)^{-2}.$$

2. To residuate $\dfrac{1}{(x-a)(x-b)(x-c)^3}$.

Ans. The residual for $x = a$, is $(a-b)^{-1}(a-c)^{-3}$,

that for $x = b$, is $(b-a)^{-1}(b-c)^{-3}$,

that for $x = c$, is $\dfrac{(2c-a-b)^2-(c-a)(c-b)}{(c-a)^3(c-b)^3}$.

Residuation.

3. To residuate cosec. x.

Solution. We have

$$\text{cosec.}\, x = \infty,$$

whenever $$x = n\pi,$$

n being an integer, and the residual of cosec. x is

$$i\,\text{cosec.}\,(n\pi+i) = \frac{i}{\sin.\,(n\pi+i)} = \frac{1}{\cos.(n\pi+i)}$$

$$= \frac{1}{\cos.\,n\pi} = \pm 1.$$

4. To residuate tang. x.

Ans. ± 1.

5. To residuate Cosec. x. *Ans.* 1.

6. To residuate (Cosec. $x)^2$. *Ans.* 0.

7. To residuate x^{-2} cosec. x.

Ans. When $x = 0$, it is $\frac{1}{6}$;

when $x = n\pi$, it is $\pm(n\pi)^{-2}$.

8. To residuate x^{-1} cosec. x.

Ans. When $x = 0$, it is 0;

when $x = n\pi$, it is $\pm(n\pi)^{-1}$.

9. To residuate $\frac{f.z}{x-z}$ for any value x_0 of z which satisfies the equation

$$f.z = \infty.$$

Method of residuating.

Solution. Let $f._i(x_0+i)$ be infinite of the [illegible]

let

$$f.z = f.z.(z - x_1 [illegible]$$

so that $f.(x_0+i)$ may be of the zero order [illegible] residual is, by (182),

$$\frac{d_{c.i}^{m-1}.f.(x_0+i).(x-x_1 [illegible]}{1 \;.\; 2 \;.\; 3 \;.\;.\;.\;. \;(m [illegible]}$$

$$= \frac{d_{c.i}^{m-1}.f.(x_0+i)[(x-x [illegible]}{1 \;.\; 2 \;.\; 3 \;.\;.\;.\;. [illegible]}$$

$$= \frac{1}{1\;.\;2\;.\;3\;.\;.\;.\;.\;m [illegible]}\Big([illegible]$$

$$+ \frac{d_{c.i}^{m-1}.f.(x_0 [illegible]}{(x-x_1 [illegible]}$$

But it is evident, from [illegible]

$$\frac{d [illegible]}{1\;.\;2\;.\;3 [illegible]}$$

is the coefficient of [illegible]

or, dividing by [illegible] development of [illegible]

$$\frac{d_{c.i}^{m-1}.f.(x_1 [illegible]}{1.2.3\ldots m [illegible]}$$

which, substituting [illegible]

$$\frac{d_x^{m-1}f.x_1 [illegible]}{1.2.3\ldots m [illegible]}$$

$$+ \frac{d [illegible]}{1\;.\;2 [illegible]}$$

3. To residuate cosec. x.

Solution. We have

$$\text{cosec. } x = \infty,$$

whenever $$x = n\pi,$$

n being an integer, and the residual of cosec. x is

$$i \text{ cosec. } (n\pi + i) = \frac{i}{\sin. (n\pi + i)} = \frac{1}{\cos. (n\pi + i)}$$

$$= \frac{1}{\cos. n\pi} = \pm 1.$$

4. To residuate tang. x.

Ans. ± 1.

5. To residuate Cosec. x. *Ans.* 1.

6. To residuate (Cosec. $x)^2$. *Ans.* 0.

7. To residuate x^{-2} cosec. x.

Ans. When $x = 0$, it is $\frac{1}{6}$;
when $x = n\pi$, it is $\pm (n\pi)^{-2}$.

8. To residuate x^{-1} cosec. x.

Ans. When $x = 0$, it is 0;
when $x = n\pi$, it is $\pm (n\pi)^{-1}$.

9. To residuate $\frac{f.z}{x-z}$ for any value x_0 of z which satisfies the equation

$$f.z = \infty.$$

Method of residuating.

Solution. Let $f.\vdots(x_0+i)$ be infinite of the mth order, and let

$$\mathrm{f}.z = f.z.(z-x_0)^m, \tag{183}$$

so that $\mathrm{f}.(x_0+i)$ may be of the zero order, and the required residual is, by (182),

$$\frac{d_{c.i}^{m-1}.\,\mathrm{f}.(x_0+i).(x-x_0-i)^{-1}}{1\,.\,2\,.\,3\,.\,.\,.\,.\,(m-1)}$$

$$=\frac{d_{c.i}^{m-1}.\mathrm{f}.(x_0+i)[(x-x_0)^{-1}+(x-x_0)^{-2}i+(x-x_0)^{-3}i^2+\&c.]}{1\,.\,2\,.\,3\,.\,.\,.\,.\,(m-1)}$$

$$=\frac{1}{1\,.\,2\,.\,3\,.\,.\,.\,.\,m-1}\left(\frac{d_{c.i}^{m-1}\,\mathrm{f}.(x_0+i)}{x-x_0}\right.$$

$$\left.+\frac{d_{c.i}^{m-1}.\mathrm{f}.(x_0+i).i}{(x-x_0)^2}+\frac{d_{c.i}^{m-1}\,\mathrm{f}.(x_0+i).i^2}{(x-x_0)^3}+\&c.\right) \tag{184}$$

But it is evident, from MacLaurin's Theorem, that

$$\frac{d_{c.i}^{m-1}.\,\mathrm{f}.(x_0+i).i^n}{1.2.3\ldots(m-1)} \tag{185}$$

is the coefficient of i^{m-1} in the development of

$$\mathrm{f}.(x_0+i).i^n \tag{186}$$

or, dividing by i^n, that (185) is the coefficient of i^{m-n-1} in the development of $\mathrm{f}.(x_0+i)$. Hence, by this theorem,

$$\frac{d_{c.i}^{m-1}.\mathrm{f}.(x_0+i)\,i^n}{1.2.3\ldots(m-1)}=\frac{d_{c.i}^{m-n-1}.\,\mathrm{f}.(x_0+i)}{1.2.3\ldots.(m-n-1)}=\frac{d_{c.}^{m-n-1}\,\mathrm{f}.(x_0)}{1.2.3\ldots(m\text{-}n\text{-}1)} \tag{187}$$

which, substituted in (184), gives, for the required residual,

$$\frac{d_{c.}^{m-1}\,\mathrm{f}.x_0}{1.2.3\ldots(m-1)}\cdot\frac{1}{x-x_0}+\frac{d_{c.}^{m-2}\,\mathrm{f}.x_0}{1.2.3\ldots(m-2)}\cdot\frac{1}{(x-x_0)^2}+\&c.\ldots.$$

$$+\frac{d_{c.}^2\,\mathrm{f}.x_0}{1\,.\,2}\cdot\frac{1}{(x-x_0)^{m-2}}+\frac{d_{c.}\,\mathrm{f}.x_0}{(x-x_0)^{m-1}}+\frac{\mathrm{f}.x_0}{(x-x_0)^m} \tag{188}$$

Integral residual.

The value of $\mathrm{f}.x_0$ is found by the equation (183), which by (521 of Vol. I.) gives

$$\mathrm{f}.x_0 = \frac{d_{c.}^m (z-x_0)^m}{d_{c.}^m (f.z)^{-1}} = \frac{1.2.3.....m}{d_{c.}^m (f.x_0)^{-1}}. \qquad (189)$$

10. To residuate the preceding example, when

$$f.z = \frac{z^2+ab}{(z-a)(z-b)^2}.$$

Ans. When $z=a$, the residual is $\frac{a^2+ab}{(a-b)^2}.\frac{1}{x-a}$;

when $z=b$, it is $\frac{b^2-2ab-a^2}{(b-a)^2}.\frac{1}{x-b}+\frac{b^2+ab}{b-a}.\frac{1}{(x-b)^2}$.

11. To residuate example 9, when

$$f.z = \text{cosec.}\, z.$$

Ans. When $z=n\pi$, the residual is $\pm\frac{1}{x-n\pi}$.

5. The *integral residual* of a function between certain limits is the sum of all its residuals contained between those limits; and the *total residual* is the sum of all its residuals.

To *residuate from one value of a variable to another* is to find the integral residual between these values of the variable; and to *residuate totally* is to find the total residual.

a. The total residual is indicated by the sign $\mathcal{E}$, and the

Notation.

integral residual is denoted by the same sign with letters annexed to it, to show the limits of the residuation; thus

$$\mathcal{E} \cdot (f \cdot x) \qquad (190)$$

is the total residual of $f.x$; while

$$\mathcal{E}_{x_0}^{x_1} \cdot (f \cdot x) \qquad (191)$$

is the integral residual of $f.x$ from the limit

$$x = x_0 \quad \text{to} \quad x = x_1.$$

b. The residuation is often limited to those values of the variable, which render one of the terms or factors of the given function infinite, as in Example 9 of the preceding section; and this is indicated by placing, in double parentheses, the factor which is thus regarded exclusively of the other factors.

Thus $$\mathcal{E} \cdot ((f \cdot x)) \cdot (f' \cdot x) \qquad (192)$$

indicates the residual of $(f.x)(f'.x)$ with regard to those values of x, which render $f.x$ infinite. In this way

$$\mathcal{E} \cdot ((f \cdot x)) \qquad (193)$$

should be used instead of (190) to denote the total residual of $f.x$. In the same way

$$\mathcal{E} \cdot \frac{f.x.(x - x_0)}{((x - x_0))} \qquad (194)$$

denotes the simple residual of

$$\frac{f.x.(x - x_0)}{x - x_0} = f.x$$

for the value of x, $\qquad x = x_0.$

Notation.

6. The variable in (191) may be itself a function of other variables, as y, z, &c.; and the residuation may be sought between the limiting values of y

$$y = y_0 \text{ and } y = y_1,$$

and those of z

$$z = z_0 \text{ and } z = z_1, \text{ \&c.}$$

and this may be expressed by the form

$$\mathcal{E}_{y = y_0,\, z = z_0,\, \text{\&c.}}^{y = y_1,\, z = z_1,\, \text{\&c.}} \cdot ((f \cdot x)), \tag{195}$$

or more simply

$$\mathcal{E}_{y_0,\, z_0,\, \text{\&c.}}^{y_1,\, z_1,\, \text{\&c.}} \cdot ((f \cdot x)), \tag{196}$$

it being conventional in what order the limits are placed.

7. *Corollary.* The preceding notation gives at once, if x' is a value of x between x_0 and x_1,

$$\mathcal{E}_{x_0}^{x_1} \cdot ((f \cdot x)) = \mathcal{E}_{x_0}^{x'} \cdot ((f \cdot x)) + \mathcal{E}_{x'}^{x_1} ((f \cdot x)). \tag{197}$$

8. *Scholium.* If x' is a root of the equation

$$f \cdot x = \infty; \tag{198}$$

the value of the corresponding residual should be equally divided between the two terms of the second members of (197), that is, when one of the limits of (191) is a root of (198), one half of the corresponding residual should be included in the expression (191).

9. *Corollary.* If, in (196), there are only two variables y and z, and if y is taken to denote the real term of x reduced

to the form (1), and z the real factor of the imaginary term, (196) will denote the integral residual for all values of x, whose real terms are included between y_0 and y_1, and the real factors of whose imaginary terms are included between z_0 and z_1.

10. *Corollary.* It is evident from (182) and § 5, *that the residual is a linear function;* and found, as it is, by differentiation, it must by § 52 of B. II. be *free relatively to any other linear function, such as difference, differential,* &c.

Thus, if the residuation is taken relatively to x, we have

$$\mathcal{L}.((d^{c.x}.f.(x,z))) = d^{c.x}.\mathcal{L}((f.(x,z))). \qquad (199)$$

CHAPTER II.

DEVELOPMENT OF FUNCTIONS, WHICH HAVE INFINITE VALUES.

11. *Problem. To develop a function which has infinite values corresponding to finite values of its variable, in a form which may be used for all values of its variable.*

Solution. Let $f.x$ be the given function, and let x_0 be a value for which it becomes infinite, so that, if i is an infinitesimal, $f.(x_0+i)$ is infinite of the mth order. Then if we put

$$\mathrm{f}.\,x = f.x.(x - x_0)^m; \tag{200}$$

we have $\mathrm{f}.\,x_0$ finite, and (200) can be developed according to powers of $x - x_0$. We have, by Taylor's Theorem,

$$\mathrm{f}.x = \mathrm{f}.x_0 + d_c.\mathrm{f}.x_0.(x-x_0) + \&\text{c.} \ldots + \frac{d_c^{m-1}.\mathrm{f}.x_0}{1.2.3\ldots(m-1)}(x-x_0)^{m-1}$$
$$+ \frac{d_c^m.\mathrm{f}.x_0}{1.2.3\ldots m}(x-x_0)^m + \frac{d_c^{m+1}.\mathrm{f}.x_0}{1.2.3\ldots(m+1)}(x-x_0)^{m+1} + \&\text{c.} \tag{201}$$

whence, by (200),

$$f.x = \frac{\mathrm{f}.x_0}{(x-x_0)^m} + \frac{d_c.\mathrm{f}.x_0}{(x-x_0)^{m-1}} + \&\text{c.} \ldots + \frac{d_c^{m-1}.\mathrm{f}.x_0}{1.2.3\ldots(m-1)} \cdot \frac{1}{x-x_0}$$
$$+ \frac{d_c^m.\mathrm{f}.x_0}{1.2.3\ldots m} + \frac{d_c^{m+1}.\mathrm{f}.x_0}{1.2.3\ldots(m+1)}(x-x_0) + \&\text{c.} \tag{202}$$

Now the upper line of the second member of (202) consists of

Function, which is always finite, when the variable is so.

terms divided by different powers $x - x_0$, all of which are finite, unless

$$x = x_0, \tag{203}$$

in which case they are infinite; while the lower line is a function of x, which is finite in this case. We will denote the upper line by X_0 and the lower line by Y_0; and X_0 is, by (188), the residual of

$$\frac{f.z}{x - z} \tag{204}$$

when $$z = x_0. \tag{205}$$

If, then, we denote by Z_0 all the other residuals of (204), when $f.z$ is infinite; we have, for the total residual of (204),

$$\mathcal{E}.\frac{((f.z))}{x - z} = X_0 + Z_0. \tag{206}$$

But by (202) $$f.x = X_0 + Y_0; \tag{207}$$

and therefore $$f.x - \mathcal{E}.\frac{((f.z))}{x - z} = Y_0 - Z_0. \tag{208}$$

Now Y_0 and Z_0 are both such functions of x that they are finite when

$$x = x_0; \tag{209}$$

that is, the first member of (208) is a function of x, which is finite for every finite value of x, such as (209), for which $f.x$ is infinite, and if we denote this function by $\varpi.x$, we have

$$f.x - \mathcal{E}.\frac{((f.z))}{x - z} = \varpi.x. \tag{210}$$

Hence the second term of (210) is finite for all finite values of x for which $f.x$ is finite; and, therefore, *$\varpi\ x$ must be finite for every finite value of x.*

Hence in the equation

$$f.x = \mathcal{E}.\frac{((f.z))}{x-z} + \varpi.x, \tag{211}$$

the first term of the second member consists, as in (188), of a combination of terms arranged according to the negative powers of $x - x_0$, $x - x_1$, &c., while $\varpi.x$ is always finite, and can usually be developed according to powers of x by Taylor's Theorem, or by some other simple process.

12. *Corollary.* When the modulus of x is infinite, the first term of the second member of (211) vanishes, and (211) becomes

$$f.\infty = \varpi.\infty. \tag{212}$$

13. *Corollary.* When the first member of (212) is finite for all values of the argument of x, $\varpi.x$ is always finite. But it has been shown, in § 31 of B. III., that the equation

$$\frac{1}{\varpi.x} = 0, \quad \text{or} \quad \varpi.x = \infty, \tag{213}$$

is always possible, unless $\varpi.x$ is constant, that is, independent of x; and, therefore, if we put

$$f.\infty = F; \tag{214}$$

we have

$$\varpi.x = F, \tag{215}$$

and in this case

$$f.x = \mathcal{E}.\frac{((f.z))}{x-z} + F. \tag{216}$$

Development of a rational fraction.

14. *Corollary.* When $f.x$ is a rational fraction, $\varpi.x$ is also a similar rational fraction, because the second term of (210) consists of the sum of such fractions. But $\varpi.x$ cannot have an entire polynomial for its denominator, because such a denominator would vanish for finite values of x, and $\varpi.x$ would become infinite. Its denominator must then be constant; that is, $\varpi.x$ must be an integral polynomial.

15. *Corollary.* If, in the preceding corollary, the degree of the numerator of $f.x$ is greater than that of its denominator, this function is infinite when its variable is infinite; but if the degree of the numerator is equal to that of the denominator, $f.x$ is finite when its variable is infinite; but if the degree oı the numerator is less than that of the denominator, $f.x$ vanishes when its variable is infinite. For if the function is

$$f.x = \frac{a x^n + b x^{n-1} + \&c.}{a' x^{n'} + b' x^{n'-1} + \&c.} \tag{217}$$

we have $$f.\infty = \frac{a\,(\infty)^n}{a'\,(\infty)^{n'}} = \frac{a}{a'}\,(\infty)^{n-n'} \tag{218}$$

which is infinite, when $n > n'$,

finite and $= \dfrac{a}{a'} = F$, when $n = n'$, (219)

zero, when $n < n'$.

The polynomial $\varpi.x$ is, therefore, reduced to a constant in the second case, and to zero in the third case.

16. *Corollary.* The easiest way of finding $\varpi.x$ in the case of § 14, is to reduce the given fraction by division to a mixed expression, consisting of an integral polynomial, and a fraction in which the degree of the numerator is less than that of the

denominator. For this last fraction can, by the preceding corollary, furnish no part of the polynomial $\varpi . x$, which must, therefore, be the same with the polynomial thus obtained by division.

17. Examples.

1. Develop $(\sin. x)^{-1}$ by the preceding principles.

Solution. The general expression for the root of the equation

$$(\sin. x)^{-1} = \infty, \tag{220}$$

is

$$x = \pm n \pi, \tag{221}$$

in which n is any integer at pleasure; and the corresponding value of the residual of

$$\frac{(\sin. z)^{-1}}{x - z}$$

is, by Ex. 9, § 4, if we put

$$\text{f}. z = \frac{1}{d_c . \sin. z} = \frac{1}{\cos. z}$$

$$\frac{1}{\cos. n\pi} \cdot \frac{1}{x \mp n\pi} \tag{222}$$

so that by (216)

$$\text{cosec}. x = \frac{1}{\sin. x} = \frac{1}{x} - \frac{1}{x + \pi} - \frac{1}{x - \pi} + \frac{1}{x + 2\pi} + \frac{1}{x - 2\pi}$$

$$= \frac{1}{x} + 2x \left(\frac{1}{\pi^2 - x^2} - \frac{1}{4\pi^2 - x^2} + \frac{1}{9\pi^2 - x^2} - \&c. \right) \tag{223}$$

Development of secants.

2. Develop $\sec. x$ by the preceding principles.

Ans. $$\sec. x = 4\pi\left(\frac{1}{\pi^2 - 4x^2} - \frac{3}{9\pi^2 - 4x^2} + \frac{5}{25\pi^2 - 4x^2} - \&c.\right) \tag{224}$$

3. Develop $(e^x + e^{-x})^{-1} = \frac{1}{2}\,\text{Sec.}\, x$.

Solution. Let $$x = y + z\sqrt{-1}, \tag{225}$$

and we have, by (89),

$$e^x = e^y (\cos. z + \sqrt{-1} \,.\, \sin. z), \tag{226}$$

$$e^{-x} = e^{-y} (\cos. z - \sqrt{-1} \,.\, \sin. z). \tag{227}$$

Hence the equation $$e^x + e^{-x} = 0, \tag{228}$$

involves the two $$(e^y + e^{-y}) \cos. z = 0, \tag{229}$$

$$(e^y - e^{-y}) \sin. z = 0. \tag{230}$$

Hence, $\cos. z = 0$, $e^y = e^{-y}$, or $e^{2y} = 1$, $y = 0$; (231)

$$z = \pm (n + \tfrac{1}{2})\,\pi, \tag{232}$$

and the root of (228) is

$$x_0 = \pm (n + \tfrac{1}{2})\,\pi \sqrt{-1}. \tag{233}$$

If, now, we take

$$\text{f}. x = \frac{1}{d_c.(e^x + e^{-x})} = \frac{1}{e^x - e^{-x}}; \tag{234}$$

we have, by (90 and 92),

$$\text{f}. x_0 = \pm \frac{1}{2\sqrt{-1}}; \tag{235}$$

and the corresponding residual of

$$\frac{(e^x + e^{-x})^{-1}}{x - z} \tag{236}$$

is $\pm\frac{1}{2\sqrt{-1}}\cdot\frac{1}{x\mp(n+\frac{1}{2})\pi.\sqrt{-1}}=\pm\frac{1}{2x\sqrt{-1}\pm(2n+1)\pi}.$ (237)

we have, then,

$$\frac{1}{e^x+e^{-x}}=\left(\frac{1}{2x\sqrt{-1}+\pi}-\frac{1}{2x\sqrt{-1}-\pi}\right)$$

$$-\left(\frac{1}{2x\sqrt{-1}+3\pi}-\frac{1}{2x\sqrt{-1}-3\pi}\right)+\text{\&c.}$$

$$=2\pi\left(\frac{1}{\pi^2+4x^2}-\frac{3}{9\pi^2+4x^2}+\frac{5}{25\pi^2+4x^2}-\text{\&c.}\right).\quad(238)$$

4. Develop $(e^x-e^{-x})^{-1}=\frac{1}{2}$ Cosec. x.

Ans. $\frac{1}{2x}-x\left(\frac{1}{\pi^2+x^2}-\frac{1}{4\pi^2+x^2}+\frac{1}{9\pi^2+x^2}-\text{\&c.}\right).$ (239)

5 Develop $\frac{x^5+1}{(x-1)^2(x+2)}$.

Solution. Since

$$(x-1)^2(x+2)=x^3-3x+2,\quad(240)$$

we have, by division,

$$\frac{x^5+1}{(x-1)^2(x+2)}=x^2+3+\frac{-2x^2+9x-5}{(x-1)^2(x+2)}.\quad(241)$$

Now by Ex. 9 of § 4

$$\mathcal{E}\cdot\left(\left(\frac{-2z^2+9z-5}{(z-1)^2(z+2)}\right)\right)\frac{1}{x-z}=\frac{2}{3(x-1)^2}$$

$$+\frac{13}{9(x-1)}-\frac{31}{9(x+2)};\quad(242)$$

whence, by (216),

$$\frac{x^5+1}{(x-1)^2\,(x+2)} = x^2+3+\frac{2}{3\,(x-1)^2}$$

$$+\frac{13}{9\,(x-1)}-\frac{31}{9\,(x+2)}. \tag{243}$$

6. Develop $\dfrac{f.x}{(x-x_0)\,(x-x_1)\,(x-x_2)\ldots\ldots}$, in which x_0, x_1, &c. are all unequal, and the values of x, which render $f.x$ infinite, are to be neglected.

Solution. We have at once

$$\frac{f.x}{(x-x_0)\,(x-x_1)\ldots} = \mathcal{E}.\frac{f.z}{(((z-x_0)\,(z-x_1)\ldots))}.\frac{1}{x-z} \tag{244}$$

$$=\frac{f.x_0}{(x_0-x_1)\,(x_0-x_2)\ldots}.\frac{1}{x-x_0}$$

$$+\frac{f.x_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots}.\frac{1}{x-x_1}+\text{\&c.}$$

7. Develop $\dfrac{x-1}{(x+1)\,(x-2)}$.

Ans. $\dfrac{2}{3(x+1)}+\dfrac{1}{3(x-2)}$.

8. Develop $\dfrac{x^4-3\,x^3+2}{(x+1)\,(x-2)^2}$.

Ans. $x-\dfrac{2}{(x-2)^2}-\dfrac{2}{3(x-2)}+\dfrac{2}{3\,(x+1)}$.

BOOK V.

INTEGRAL CALCULUS.

BOOK V.

INTEGRAL CALCULUS.

CHAPTER I.

INTEGRATION.

1. The *integral of a given differential* is the function of which it is the differential; and *the integral of a given finite function* is the function of which it is the differential coefficient.

To integrate is to find the integral. The sign of integration is $\int.$; thus

$$\int . d . x = x, \quad \int . d . f . x = f . x;$$

$$\int . d_{c} . x = x, \quad \int . d_{c} . f . x = f . x; \qquad (245)$$

$$\int^{2} . d^{2}_{c} . x = x, \quad \int^{n} . d^{n}_{c} . f . x = f . x, \text{ \&c.} \qquad (246)$$

2. *Corollary.* Since we have

$$d_{c} . (x + a) = d_{c} . f . x, \qquad (247)$$

for all values of a, it follows that

$$\int . d_{c} . f . x = f . x + a, \qquad (248)$$

that is, the integral of a function may have *an arbitra-*

ry constant added to it, and in this form the integral is said to be *complete*.

3. *Corollary*. Any constant may then be added to, or subtracted from the incomplete integral, and the form of the integral may often be changed by this process.

4. *Corollary*. If the integral contains a term of the form

$$\log. f\ x,$$

this term may be changed, by the addition of a constant, to the form

$$\log. f. x + \log. a = \log. (a f. x). \qquad (249)$$

5. *Corollary*. If the integral contains a term of the form

$$\sin.^{[-1]} x,$$

this term may be changed, by the addition of a constant, to the form

$$\sin.^{[-1]} x - \tfrac{1}{2}\pi = -(\tfrac{1}{2}\pi - \sin.^{[-1]} x) = -\cos.^{[-1]} x \quad (250)$$

or it may be changed into

$$\text{cose.}^{[-1]} \frac{1}{x}, \text{ or into } \cos.^{[-1]} \sqrt{(1-x^2)} \text{ or into } -\sin.^{[-1]} \sqrt{(1-x^2)}.$$

In the same way, terms of the form

$$\cos.^{[-1]} x, \quad \tan.^{[-1]} x, \quad \cot.^{[-1]} x, \quad \sec.^{[-1]} x, \ \&c.$$

may be changed into

$$-\sin.^{[-1]} x, \quad \cot.^{[-1]} x, \quad -\tan.^{[-1]} x, \quad -\text{cosec.}^{[-1]} x, \ \&c.$$

or into

$$\sec.^{[-1]} \frac{1}{x}, \quad -\tan.^{[-1]} \frac{1}{x}, \quad \tan.^{[-1]} \frac{1}{x}, \quad \cos.^{[-1]} \frac{1}{x}, \ \&c.$$

Number of arbitrary constants. Definite integral.

or into (251)

$$\sin.^{[-1]}\sqrt{(1-x^2)},\ \sin.^{[-1]}\frac{x}{\sqrt{(1+x^2)}},\ \sin.^{[-1]}\frac{1}{\sqrt{(1+x^2)}},$$

$$\sin.^{[-1]}\frac{1}{\sqrt{(1-x^2)}},\ \&c.$$

6. *Corollary.* Since every integration introduces an arbitrary constant, the number of arbitrary constants in a complete integral must be equal to the number of integrations.

7. *Corollary.* The difference between the two values of an integral, which correspond to two values of its variable, is called *the definite integral from one value to the other value of the variable.*

Thus if x_0 and x_1 are the limiting values of the variable, the integral of $d_e . f . x$ from x_0 to x_1 is, by (248),

$$(f . x_1 + a) - (f . x_0 + a) = f . x_1 - f . x_0\,; \qquad (252)$$

and it is written

$$\int_{x_0}^{x_1} . d_e . f . x = f . x_0 . \qquad (252)$$

The definite integral is, therefore, *independent of the value of the arbitrary constant;* but the places of the arbitrary constant and the variable are supplied by regarding one of the limits as arbitrary and the other as variable, thus

$$\int_{x_0}^{x} . d_e . f . x = f . x - f . x_0, \qquad (254)$$

which gives, by (248),

$$a = -f . x_0 . \qquad (255)$$

8. *Corollary.* Since

$$\int_{x_1}^{x_0} . d_c . f . x = f . x_0 - f . x_1, \tag{256}$$

we have, obviously,

$$\int_{x_0}^{x_1} . = - \int_{x_1}^{x_0} . \tag{257}$$

9. *Corollary.* Equation (246) shows that integration may be regarded as negative differentiation, that is,

$$d_c^{-n} = \int^n \tag{258}$$

10. *Corollary.* It is evident, from B. II. §§ 51 and 52, that *integrals are linear functions, which are free relatively to all other linear functions.*

Thus we have $\int . a f . x = a \int . f . x.$ (259)

11. *Corollary. Differentials, residuals, and integrals are functions which are relatively free.*

12. *Corollary.* When a function can be separated into parts connected by the signs + or —, *the integral of the whole function is the algebraic sum of the partial integrals.*

This method of integration might naturally be called integration by parts, but the following is a particular case of it, to which this designation has been applied technically.

13. If u and v are functions of a variable, we have (Vol. I. 468)

$$d_c . u v = u \, d_c . v + v \, d_c . u, \tag{260}$$

Integration by parts.

whence $$u d_e . v = d_e . u v - v d_e . u, \tag{261}$$

and by integration

$$f . u d_e . v = u v - f . v d_e . u ; \tag{262}$$

and when a given differential coefficient can be separated into two factors, one of which, $d_e . v$, has a known integral, the integration can often be effected by the aid of (262); and the application of this formula is called *integration by parts.*

14. *Theorem.* A definite integral, which is taken between limits differing by a quantity equal to the differential of the variable, is equal to the differential of the integral.

Proof. For the equation (252) becomes, when

$$x_0 = x, \qquad x_1 = x + d x, \tag{263}$$

by (Vol. I. 421)

$$\int_x^{x+dx} . d_e . f . x = f . (x + d x) - f . x = d . f . x. \tag{264}$$

15. *Theorem.* If x_0, x_1, x_2, . . . ; x_n are successive values of x, a definite integral from x_0 to x_n, is equal to the algebraic sum of the corresponding definite integrals from x_0 to x_1, from x_1 to x_2, &c.

Proof. We evidently have

$$f . x_n - f . x_0 = (f . x_1 - f . x_0) + (f . x_2 - f . x_1) \\ + (f . x_3 - f . x_2) + \&c. \tag{265}$$

16. *Corollary.* Hence if x_0, x_1, x_2, &c. differ by $d x$,

the definite integral from x_0 to x_n is equal to the algebraic sum of all the corresponding differentials from x_0 to x_n, taken at intervals equal to dx.

17. *Scholium.* Propositions 14 and 16 require that the integral be a continuous function between the limits, and particular caution must be observed to exclude those cases, in which the value of the integral varies from positive to negative, or the reverse, by passing through infinity, so as suddenly to vary from positive to negative infinity, or the reverse.

18. *Theorem.* If we have the equation

$$\int . f . x = F . x \tag{266}$$

and if we substitute for x any function at pleasure, as $\varphi . x$, we shall have

$$\int . f . (\varphi . x) . d_c . \varphi . x = F . \varphi . x. \tag{267}$$

Proof. For (266) gives by differentiation

$$d_c . F . x = f . x, \tag{268}$$

and putting

$$x = \varphi . y \tag{269}$$

we have, by (Vol. I. 566),

$$d_{c.y} F . \varphi . y = f . (\varphi . y) . d_c . \varphi . y, \tag{270}$$

and by integration

$$F . \varphi . y = \int . f . (\varphi . y) . d_c . \varphi . y, \tag{271}$$

which is the same with (267), changing y to x.

19. *Corollary.* When $x = \varphi . y$ (272)

we have $\int . f . x = \int . f . (\varphi . y) . d_c . \varphi . y.$ (273)

CHAPTER II.

INTEGRATION OF RATIONAL FUNCTIONS.

20. *Problem. To integrate an algebraic monomial.*

Solution. First. If the algebraic monomial is

$$a x^n, \tag{274}$$

in which n differs from -1, the substitution in 262 of

$$u = a x^n, \quad v = x, \tag{275}$$

$$d_c . u = n a x^{n-1}, \quad d_c v = 1, \tag{276}$$

gives $\int . a x^n = a x^{n+1} - \int . n a x^n = a x^{n+1} - n \int . a x^n$; (277)

whence, by transposition and division,

$$n \int . a x^n + \int . a x^n = (n+1) \int . a x^n = a x^{n+1}, \tag{278}$$

$$\int . a x^n = \frac{a x^{n+1}}{n+1}; \tag{279}$$

that is, *the integral of the monomial is found, by increasing the exponent of the variable by unity, and dividing by the exponent thus increased.*

Secondly. An arbitrary constant should be added to (279) for the complete integral, and we have as in (254)

$$\int_{x_0}^{x} . a x^n = \frac{a (x^{n+1} - x_0^{n+1})}{n+1} \tag{280}$$

Thirdly. When $n = -1$

(279) becomes infinite. But the infinite form may be avoided by means of an infinite arbitrary constant, such as that of (280). In this case, (280) assumes an indeterminate form, the true value of which may be ascertained by means of B. II. § 104. For the differentiation of the terms of (280) relatively to n gives, by (Vol. I. 481),

$$\begin{aligned} \int_{x_0}^{x} . a x^{-1} &= \frac{a\, d_{c \cdot n} . (x^{n+1} - x_0^{n+1})}{d_{c \cdot n} . (n+1)} \\ &= a\,(x^{n+1} \log. x - x_0^{n+1} \log. x_0) \\ &= a\,(\log. x - \log. x_0), \end{aligned} \tag{281}$$

or omitting the arbitrary constant $a \log. x_0$

$$\int . \frac{a}{x} = a \log. x. \tag{282}$$

21. *Corollary. Every algebraic polynomial, being the sum of monomials of the form* (274), *may be integrated by integrating its terms separately; and any function can also be integrated by this process, which can be reduced to such a polynomial.*

22. Examples.

1. Integrate $6\,x^5 + x^{\frac{1}{2}} + 5 \sqrt[3]{x^2} - \frac{1}{x^5} + 8\,x^{-9}$.

Ans. $x^6 + \frac{2}{3} x^{\frac{3}{2}} + 3\,x^{\frac{5}{3}} + \frac{1}{4} x^{-4} - x^{-8}$.

2. Integrate $3 \sqrt{x} + \frac{1}{2} x^{-\frac{1}{2}}$. *Ans.* $2\,x^{\frac{3}{2}} + \sqrt{x}$.

3. Integrate $a x^3 + b x^2 + \frac{c}{x} + h.$

Ans. $\frac{1}{4} a x^4 + \frac{1}{3} b x^3 + c \log. x + h x.$

4. Integrate $x^{-2} (x^2 + x + 1)^2.$

Ans. $\frac{1}{3} x^3 + x^2 + 3 x + 2 \log. x - x^{-1}.$

5. Integrate $x (x + x^{-1})^2.$ *Ans.* $\frac{1}{4} x^4 + x^2 + \log. x.$

23. *Corollary.* The substitution of $\varphi . x$ for x in (279) and (282), gives by § 18,

$$\int . a (\varphi . x)^n d_c . \varphi . x = \frac{a (\varphi . x)^{n+1}}{n+1}; \tag{283}$$

$$\int . \frac{a\, d_c . \varphi . x}{\varphi . x} = a \log. \varphi . x. \tag{284}$$

24. *Corollary.* Let

$$\varphi . x = b x + c, \quad d_c . \varphi . x = b, \tag{285}$$

(283 and 284) become, by dividing by b,

$$\int . a (b x + c)^n = \frac{a (b x + c)^{n+1}}{b (n+1)}; \tag{286}$$

$$\int . \frac{a}{b x + c} = \frac{a}{b} \log. (b x + c.) \tag{287}$$

25. ***Problem.*** *To integrate a rational fraction.*

Solution. Let the fraction be reduced, as in B. IV. § 16, to a mixed quantity, of which one part is an integral polynomial, and the other is a rational fraction, in which the degree of the

numerator is less than that of the denominator. If this second part is denoted by $f.x$, (216) and B. IV. § 15 give

$$f.x = \Sigma.\frac{((f.z))}{x-z}. \qquad (288)$$

The integral of (288), relatively to x, is by (287) and § 11

$$\mathit{f}.f.x = \Sigma.\int\frac{((f.z))}{x-z} = \Sigma.((f.z))\log.(x-z); \qquad (289)$$

and (289), added to the integral of the polynomial, is the required integral.

26. *Corollary.* Since (288) is, by the process of B. IV. § 4, Ex. 9, reduced to the sum of several fractions of the form

$$\frac{f'.x_0}{(x-x_0)^n}, \qquad (290)$$

(289) is the sum of their integrals, or is itself by (286 and 287) the sum of several terms of the form

$$-\frac{f.'x_0}{(n-1)(x-x_0)^{n-1}} \qquad (291)$$

when n is greater than unity, and of the form

$$f.x_0\log.(x-x_0) \qquad (292)$$

when n is unity.

27. *Corollary.* When the given rational fraction is a real function, it follows from B. III. §§ 37–39, that x_0', the conjugate of x_0, furnishes a fraction, corresponding to (290)

$$\frac{f'.x_0'}{(x-x_0')^n}; \qquad (293)$$

such that if $$f.' x_0 = A + B\sqrt{-1}, \tag{294}$$

then $$f.' x_0' = A - B\sqrt{-1}; \tag{295}$$

and if we put $$(x - x_0)^{n-1} = X + Y\sqrt{-1}, \tag{296}$$

in which X and Y are functions of x, we have

$$(x - x_0')^{n-1} = X - Y\sqrt{-1}. \tag{297}$$

Hence the sum of (291) and the conjugate fraction is, supposing

$$x_0 = a + b\sqrt{-1}, \tag{298}$$

$$\frac{1}{n-1} \cdot \frac{(A+B\sqrt{-1})(X-Y\sqrt{-1}) + (A-B\sqrt{-1})(X+Y\sqrt{-1})}{(x - x_0)^{n-1}\ (x - x_0')^{n-1}}$$

$$= -\frac{1}{n-1}\ \frac{2AX + 2BY}{(x^2 - 2ax + a^2 + b^2)^{n-1}}, \tag{299}$$

which is a real function.

In the same way, since (95) gives

$$\log.(x - x_0) = \log.(x - a - b\sqrt{-1})$$

$$= \tfrac{1}{2}\log.[(x-a)^2 + b^2] - \tan.^{[-1]}\frac{b}{x-a}.\sqrt{-1}, \tag{300}$$

the sum of (292) and its conjugate is

$$A\log.[(x - a)^2 + b^2] + 2B\tan.^{[-1]}\frac{b}{x-a} \tag{301}$$

which is real; so that the required integral is thus entirely freed from imaginary quantities.

28. Examples.

1. Integrate

$$\frac{x^7 + 10x^6 + 36x^5 + 67x^4 + 68x^3 + 29x^2 - 4x - 7}{(x^2 + 2x + 2)^2 (x+1)^3 (x-1)}.$$

Solution. First. When, for this example,

$$x_0 = 1,$$

we have $\quad n = 1, \quad f'. x_0 = 1,$

and (292) becomes $\quad \log. (x - 1).$

Secondly. When $\quad x_0 = -1,$

we have for $\quad n = 3, \quad f'. x_0 = 1,$

so that (291) becomes $\quad -\dfrac{1}{2(x+1)^2};$

and we have for $\quad n = 2, \quad f'. x_0 = 0,$

for $\quad n = 1, \quad f'. x_0 = 0.$

Thirdly. When $\quad x_0 = -1 + \sqrt{-1},$

we have for $\quad n = 2, \quad f'. x_0 = -\frac{1}{4};$

so that by (294 and 295)

$$A = -\tfrac{1}{4}, \qquad B = 0,$$

$$X = x + 1, \quad Y = -1,$$

and (299) becomes $\quad \dfrac{x+1}{2(x^2+2x+2)}.$

We have also for $\quad n = 1, \quad f'. x_0 = \frac{7}{4},$

so that by (294 and 296)

Integration of rational fractions.

$$A = 0, \quad B = -\tfrac{7}{4},$$

and (301) becomes $-\tfrac{7}{4}\cot.^{[-1]}(x+1)$.

The required integral is, therefore,

$$\log.(x-1) - \tfrac{1}{2}(x+1)^{-2} + \frac{x+1}{2(x^2+2x+2)} - \tfrac{7}{4}\cot.^{[-1]}(x+1).$$

2. Integrate $\dfrac{x^5+1}{(x-1)^2(x+2)}$.

Ans. $\tfrac{1}{3}x^3 + 3x - \dfrac{2}{3(x-1)} + \tfrac{13}{9}\log.(x-1) - \tfrac{31}{9}\log.(x+2)$.

3. Integrate $\dfrac{x-1}{(x+1)(x-2)}$.

Ans. $\tfrac{2}{3}\log.(x+1) + \tfrac{1}{3}\log.(x-2)$.

4. Integrate $\dfrac{x^4-3x^3+2}{(x+1)(x-2)^2}$.

Ans. $\tfrac{1}{2}x^2 + \dfrac{2}{x-2} - \tfrac{2}{3}\log.(x-2) + \tfrac{2}{3}\log.(x+1)$.

5. Integrate $\dfrac{nx+m}{x^2-2ax+a^2+b^2}$.

Ans. $$\frac{n}{2}\log.(x^2-2ax+a^2+b^2)$$
$$+\frac{na+m}{b}\tan.^{[-1]}\frac{x-a}{b}. \qquad (302)$$

6. Integrate $\dfrac{nx+m}{ax^2+b}$.

Ans. $$\frac{n}{2a}\log.\left(x^2+\frac{b}{a}\right) - \frac{m}{\sqrt{(ab)}}\tan.^{[-1]}\frac{\sqrt{b}}{\sqrt{a}.x}. \qquad (303)$$

Integration of rational fractions.

7. Integrate $\frac{1}{x^2+1}$.

$$Ans. \quad \tan.^{[-1]} x = \cot.^{[-1]} \frac{1}{x}. \qquad (304)$$

8. Integrate $\frac{n\,x+m}{(x^2-2\,a\,x+a^2+b^2)^2}$.

$$Ans. \quad \frac{(n\,a+m)\,x-n\,(a^2+b^2)-m\,a}{2\,b^2\,(x^2-2\,a\,x+a^2+b^2)}$$

$$-\tfrac{1}{2}\,\frac{n\,a+m}{b^3}\tan.^{[-1]}\frac{b}{x-a}. \qquad (305)$$

CHAPTER III.

INTEGRATION OF IRRATIONAL FUNCTIONS.

29. *To integrate* $f.[x, \sqrt[n]{(ax+b)}]$.

Solution. Let
$$y = \sqrt[n]{(ax+b)}, \tag{306}$$
whence
$$x = \frac{y^n - b}{a},\ d_{c.y.}x = \frac{n\,y^{n-1}}{a}, \tag{307}$$
and by § 19
$$\int.f.[x, \sqrt[n]{(ax+b)}] = \int.f.\left(\frac{y^n-b}{a}, y\right) d_{c.y.}x$$
$$= \int.f.\left(\frac{y^n-b}{a}, y\right).\frac{n\,y^{n-1}}{a}. \tag{308}$$

30. Examples.

1. Integrate $\sqrt[n]{(ax+b)^m}$.

Solution. Equation (308) becomes, in this case,
$$\int.\sqrt[n]{(ax+b)^m} = \int.\frac{n\,y^{m+n-1}}{a} = \frac{n\,y^{m+n}}{(m+n)\,a}$$
$$= \frac{n\sqrt[n]{(ax+b)^{m+n}}}{(m+n)\,a}.$$

2. Integrate $\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$.

Ans. $x + 4\sqrt{x} + 4\log.(\sqrt{x} - 1)$.

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3. Integrate $\frac{\sqrt{x}+1}{\sqrt[3]{x}-1}$.

Solution. Let $y = \sqrt[6]{x}$

and (308) becomes

$$\int \frac{\sqrt{x}+1}{\sqrt[3]{x}-1} = \int \frac{y^3+1}{y^2-1} 6\, y^5$$

$$= \tfrac{6}{7} x^{\frac{7}{6}} + \tfrac{6}{5} x^{\frac{5}{6}} + \tfrac{3}{2} x^{\frac{2}{3}} + 2 x^{\frac{1}{2}} + 3 x^{\frac{1}{3}} + 6 x^{\frac{1}{6}} + 6 \log. (\sqrt[6]{x} - 1).$$

4. Integrate $x \sqrt[n]{(x+a)} + \sqrt{(x+a)}$.

Ans. $\frac{n \sqrt[n]{(x+a)^{2n+1}}}{2n+1} - \frac{n a \sqrt[n]{(x+a)^{n+1}}}{n+1} + \tfrac{2}{3} \sqrt{(x+a)^3}$.

5. Integrate $\frac{[\sqrt{(ax+b)}+c]^n}{\sqrt{(ax+b)}}$.

Ans. $\frac{2 [\sqrt{(ax+b)}+c]^{n+1}}{a(n+1)}$.

31. ***Problem.*** *To integrate* $f.[x, \sqrt{(ax^2+bx+c)}]$.

Solution. Let $x = y - \frac{b}{2a}, \; m = \frac{b^2 - 4ac}{4a^2}$, (309)

whence $a x^2 + bx + c = a y^2 - \frac{b^2 - 4ac}{4a}$

$= a(y^2 - m)$ (310)

$d_{x.y} x = 1;$ (311)

and by § 19,

$$\int .f.[x, \sqrt{(ax^2+bx+c)}] = \int .f.\left[y - \frac{b}{2a}, \sqrt{[a(y^2-m)]}\right] \quad (312)$$

There are, then, several cases:

First. When a is negative and m negative, the radical

$$\sqrt{[a(y^2 - m)]} \quad (313)$$

is always imaginary, and the integral, being imaginary, admits of no real solution, and may be solved as in either of the other cases which, in this case, become imaginary.

Secondly. When a is positive and m negative, y^2 must be greater than m, when (313) is real, let, then,

$$z = \sqrt{(y^2 - m)} - y, \quad (314)$$

whence $\quad (y + z)^2 = y^2 + 2yz + z^2 = y^2 - m \quad (315)$

$$y = -\frac{m}{2z} - \tfrac{1}{2}z \quad (316)$$

$$d_{c.z}.y = \frac{m}{2z^2} - \tfrac{1}{2} \quad (317)$$

$$\sqrt{[a(y^2 - m)} = \sqrt{a}.(z+y) = \sqrt{a}.\left(-\frac{m}{2z} + \tfrac{1}{2}z\right) \quad (318)$$

and (312) becomes

$$\int .f.\left[-\frac{m}{2z} - \tfrac{1}{2}z - \frac{b}{2a}, \sqrt{a}.\left(\tfrac{1}{2}z - \frac{m}{2z}\right)\right].\left(\frac{m}{2z^2} - \tfrac{1}{2}\right). \quad (319)$$

Thirdly. When m is positive, let

$$z^2 = \frac{a(y - \sqrt{m})}{y + \sqrt{m}}, \quad (320)$$

whence
$$y = \frac{\sqrt{m}.(z^2+a)}{a-z^2} \tag{321}$$

$$d_{c.x}.y = \frac{4\,a\sqrt{m}\,.\,z}{(a-z^2)^2} \tag{322}$$

$$\sqrt{[a(y^2-m)]} = \frac{2\,a\sqrt{m}\,.\,z}{a-z^2} \tag{323}$$

and (312) becomes

$$\int.f.\left[\frac{\sqrt{m}.(z^2+a)}{a-z^2} - \frac{b}{2\,a}, \frac{2\,a\sqrt{m}.z}{a-z^2}\right].\frac{4\,a\sqrt{m}.z}{(a-z^2)^2}. \tag{324}$$

32. Examples.

1. Integrate $\dfrac{1}{\sqrt{(a x^2+b x+c)}}$.

Solution. In this case (312) becomes

$$\int.\frac{1}{\sqrt{[a\,(y^2-m)]}} \tag{325}$$

(319) becomes, by reduction,

$$-\int.\frac{1}{\sqrt{a}\,.\,z} = -\frac{1}{\sqrt{a}}\log.\,z$$

$$= -\frac{1}{\sqrt{a}}\log.\,[\sqrt{(y^2-m)}-y],$$

or since

$$\sqrt{(y^2-m)}-y = \frac{-m}{\sqrt{(y^2-m)}+y}$$

$$\log.\sqrt{(y^2-m)}-y = \log.(-m)-\log.[\sqrt{(y^2-m)}+y]$$

$$-\int.\frac{1}{\sqrt{a}.z} = -\frac{1}{\sqrt{a}}\log.-m + \frac{1}{\sqrt{a}}\log.[\sqrt{(y^2-m)}+y], \tag{326}$$

in which $-\frac{1}{\sqrt{a}}\log. - m$ may be omitted, as in art. 3.

Again (324) becomes

$$\int \cdot \frac{2}{a - z^2}, \tag{327}$$

which, when a is positive, is

$$\frac{1}{\sqrt{a}} \cdot \log. \frac{z+\sqrt{a}}{z-\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot \log. \frac{\sqrt{(y+\sqrt{m})}+\sqrt{(y-\sqrt{m})}}{\sqrt{(y-\sqrt{m})}-\sqrt{(y+\sqrt{m})}}$$

$$= \frac{1}{\sqrt{a}} \cdot \log. [\sqrt{(y^2 - m)} + y] - \frac{\log. m}{2\sqrt{a}}; \tag{328}$$

but when a is negative (327) is as in (303)

$$\frac{2}{\sqrt{-a}} \tan.^{[-1]} \frac{\sqrt{-a}}{z} = \frac{2}{\sqrt{-a}} \tan.^{[-1]} \sqrt{} \cdot \left(\frac{\sqrt{m}-y}{\sqrt{m}+y}\right) \tag{329}$$

The form of this last solution may be changed in several ways, which will often be useful; thus, let

$$\theta = \tan.^{[-1]} \sqrt{\left(\frac{\sqrt{m} - y}{\sqrt{m} + y}\right)} \tag{330}$$

whence $\tan. \theta = \sqrt{\left(\frac{\sqrt{m} - y}{\sqrt{m} + y}\right)}$

$$\sec.^2 \theta = 1 + \tan.^2 \theta = \frac{2\sqrt{m}}{\sqrt{m} + y}$$

$$\cos.^2 \theta = \frac{\sqrt{m} + y}{2\sqrt{m}}$$

$$\sin.^2 \theta = \tan.^2 \theta \cdot \cos.^2 \theta = \frac{\sqrt{m} - y}{2\sqrt{m}}$$

$$\sin.^2 2\theta = 4 \sin.^2 \theta \cdot \cos.^2 \theta = \frac{m - y^2}{m} = \frac{4a(ax^2 + bx + c)}{4ac - b^2}$$

$$\cos. 2\theta = 2\cos.^2 \theta - 1 = \frac{y}{\sqrt{m}} = \frac{b + 2ax}{\sqrt{(b^2 - 4ac)}}$$

and (329) gives

$$\int \frac{1}{\sqrt{(ax^2+bx+c)}} = \frac{2\theta}{\sqrt{-a}} \tag{331}$$

$$= \frac{1}{\sqrt{-a}} \sin.^{[-1]} \frac{2\sqrt{-a}\,.\sqrt{(ax^2+bx+c)}}{\sqrt{(b^2-4ac)}}$$

$$= \frac{1}{\sqrt{-a}} \cos.^{[-1]} \frac{b+2ax}{\sqrt{(b^2-4ac)}}$$

$$= \pm \frac{1}{\sqrt{-a}} \sin.^{[-1]} \frac{\mp b \mp 2ax}{\sqrt{(b^2-4ac)}}.$$

2. Integrate $\frac{1}{\sqrt{(1-x^2)}}$. *Ans.* $\sin.^{[-1]} x$. (332)

3. Integrate $\frac{1}{\sqrt{(1+x^2)}}$. (333)

Ans. $\log.\, [x + \sqrt{(1+x^2)}]$.

4. Integrate $\frac{1}{\sqrt{(x^2-1)}}$. (334)

Ans. $\log.\, [x + \sqrt{(x^2-1)}]$.

5. Integrate $\frac{x^3}{\sqrt{(1-x^2)}}$.

Ans. $-(\frac{1}{3}x^2 + \frac{2}{3})\sqrt{(1-x^2)}$.

6. Integrate $\frac{1}{x\sqrt{(a+bx^2)}}$. (335)

Ans. $$\frac{1}{\sqrt{a}} \log.\, \frac{\sqrt{(a+bx^2)} - x\sqrt{b} - \sqrt{a}}{\sqrt{(a+bx^2)} - x\sqrt{b} + \sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}} \log.\, \frac{\sqrt{(a+bx^2)} - \sqrt{a}}{\sqrt{(a+bx^2)} + \sqrt{a}}$$

or $$-\frac{1}{\sqrt{-a}} \sin.^{[-1]} \frac{1}{x}\sqrt{-\frac{a}{b}}$$

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7. Integrate $\frac{1}{x\sqrt{(1+x^2)}}$. (336)

Ans. $\log. \frac{\sqrt{(1+x^2)}-1}{x}$.

8. Integrate $\frac{1}{x\sqrt{(1-x^2)}}$. (337)

Ans. $\log. \frac{\sqrt{(1-x^2)}-1}{x}$.

9. Integrate $\frac{1}{x\sqrt{(x^2-1)}}$. (338)

Ans. $-\sin.^{[-1]} \frac{1}{x}$ or $\sec.^{[-1]} x$.

10. Integrate $\frac{1}{\sqrt{(x^2+x)}}$.

Ans. $\log. [\frac{1}{2}+x+\sqrt{(x^2+x)}]$.

33. *Problem.* To integrate $\int. \left[x, \sqrt[n]{\frac{a+bx}{c+hx}}\right]$. (339)

Solution. Let

$$y^n = \frac{a+bx}{c+hx}; \tag{340}$$

whence

$$x = \frac{a-cy^n}{hy^n-b}, \tag{341}$$

$$d_{x,y}.x = \frac{n(bc-ah)y^{n-1}}{(hy^n-b)^2}, \tag{342}$$

and, by § 19, the integral of (339) becomes

$$\int.f.\left[\frac{a-cy^n}{hy^n-b}, y\right].\frac{n(bc-ah)y^{n-1}}{(hy^n-b)^2}. \tag{343}$$

34. Examples.

1. Integrate $\sqrt[3]{\frac{1-x}{1+x}}$.

Ans. $\sqrt[3]{(1-x)}.\sqrt[3]{(1+x)^2}-\log.[\sqrt[3]{(1-x)}+\sqrt[3]{(1+x)}]$

$$-\tfrac{2}{3}\sqrt{3}\tan.^{[-1]}\frac{\sqrt{3}.\sqrt[3]{(1-x)}}{2\sqrt[3]{(1+x)}-\sqrt[3]{(1-x)}}.$$

2. Integrate $\frac{1}{(1+x)^2}.\sqrt[3]{\frac{1-x}{1+x}}$.

Ans. $\tfrac{3}{8}\sqrt[3]{\left(\frac{1-x}{1+x}\right)^4}$.

35. *Problem. To integrate*

$$x^{m-1}f.[\sqrt[q]{(a+b\,x^n)},\ x^m,\ x^n], \tag{344}$$

when m is exactly divisible by n.

Solution. Let
$$a+b\,x^n=y^q; \tag{345}$$

whence
$$x^n=\frac{y^q-a}{b}, \tag{346}$$

$$x^m=\left(\frac{y^q-a}{b}\right)^{\frac{m}{n}}. \tag{347}$$

The differential coefficient of the logarithm of (346), gives

$$\frac{n\,d_{c.y}.x}{x}=\frac{q\,y^{q-1}}{y^q-a}, \tag{348}$$

whence

$$x^{m-1}d_{c.y}.x=\frac{q\,y^{q-1}}{n(y^q-a)}\left(\frac{y^q-a}{b}\right)^{\frac{m}{n}} \tag{349}$$

and, by § 19, the integral of (344) becomes

$$\int \cdot f \cdot \left[y, \left(\frac{y^q - a}{b}\right)^{\frac{m}{n}}, \frac{y^q - a}{b}\right] \cdot \frac{q\, y^{q-1}}{n(y^q - a)} \left(\frac{y^q - a}{b}\right)^{\frac{m}{n}}. \quad (350)$$

36. *Corollary.* When $q = 2$ and $2\,m$ is divisible by n, (350) is integrable by § 31.

37. Examples.

1. Integrate $x^3 \sqrt[5]{(a + b\,x^4)}$.

Solution. In this case

$$m = 4, \quad n = 4, \quad q = 5;$$

whence $$y = \sqrt[5]{(a + b\,x^4)}$$

$$\int . x^3 \sqrt[5]{(a + b\,x^4)} = \int . \frac{5\,y^5}{4\,b} = \frac{5\,y^6}{24\,b}$$

$$= \frac{5 \sqrt[5]{(a + b\,x^4)^6}}{24\,b}.$$

2. Integrate $x^3 \sqrt[5]{(a + b\,x^2)}$.

Ans. $\left(\frac{5\,x^2}{22\,b} - \frac{25\,a}{132\,b^2}\right) \sqrt[5]{(a + b\,x^2)^6}$.

3. Integrate $\frac{x^5}{\sqrt{(1 - x^2)}}$.

Ans. $-(\frac{1}{5}\,x^4 + \frac{4}{15}\,x^2 + \frac{8}{15}) \sqrt{(1 - x^2)}$.

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4. Integrate $\frac{x^2}{\sqrt{(1-x^2)}}$.

Ans. $-\frac{1}{2}x\sqrt{(1-x^2)}+\frac{1}{2}\sin.^{[-1]}x$.

38. *Problem. To integrate*

$$x^{m-1}(a+bx^n)^{\frac{p}{q}}f.(x^n) \tag{351}$$

when $\frac{m}{n}+\frac{p}{q}$ *is an nteger.*

Solution. Let
$$ax^{-n}+b=y^q, \tag{352}$$

whence
$$x^n=\frac{a}{y^q-b}, \tag{353}$$

$$a+bx^n=\frac{a\,y^q}{y^q-b}, \tag{354}$$

$$x^m=\left(\frac{a}{y^q-b}\right)^{\frac{m}{n}}. \tag{355}$$

The differential coefficient of the logarithm of (353) gives

$$\frac{n\,d_{c.y}.x}{x}=-\frac{q\,y^{q-1}}{y^q-b}, \tag{356}$$

whence

$$x^{m-1}d_{c.y}.x=-\left(\frac{a}{y^q-b}\right)^{\frac{m}{n}}\frac{q\,y^{q-1}}{n\,(y^q-b)}, \tag{357}$$

and, by § 19, the integral of (351) becomes

$$-\int.\left(\frac{a}{y^q-b}\right)^{\frac{m}{n}+\frac{p}{q}}\frac{q\,y^{p+q-1}}{n\,(y^q-b)}f.\left(\frac{a}{y^q-b}\right). \tag{358}$$

39. Examples.

1. Integrate $\dfrac{1}{x^5 \sqrt[3]{(a + b x^3)}}$.

$$Ans.\quad -\left(\frac{1}{4 a x^3} - \frac{3 b}{4 a^2}\right)\left(\frac{\sqrt[3]{(a+b x^3)^2}}{x}\right).$$

2. Integrate $\dfrac{\sqrt[5]{(a + b x^5)^3}}{x^9}$.

$$Ans.\quad -\frac{\sqrt[5]{(a + b x^5)^8}}{8 a x^8}.$$

40. ***Problem.*** *To integrate*

$$x^m (a + b x^n)^p \tag{359}$$

when m and n are positive integers, and p is a positive fraction.

Solution. ***First.*** Let
$$v = x^s, \tag{360}$$
whence
$$d_c . v = s x^{s-1}, \tag{361}$$
in which s is to be taken of such a value as may be found most useful; let, then,

$$u\, d_c . v = x^m (a + b x^n)^p; \tag{362}$$

whence
$$u = \frac{1}{s} x^{m-s+1} (a + b x^n)^p, \tag{363}$$

$$d_c . u = \frac{m-s+1}{s} x^{m-s} (a+b x^n)^p + \frac{b n p}{s} x^{m-s+n} (a+b x^n)^{p-1}; \tag{364}$$

or, since $(a+bx^n)^p = (a+bx^n)(a+bx^n)^{p-1}$, (365)

$$d_x u = \frac{(m-s+1)a+(m-s+1+np)bx^n}{s} x^{m-s}(a+bx^n)^{p-1}, \quad (366)$$

and, by (262), the integral of (359) is

$$\frac{1}{s} x^{m+1}(a+bx^n)^p$$

$$-\int . \frac{(m-s+1)a+(m-s+1+np)bx^n}{s} x^m (a+bx^n)^{p-1}, \quad (367)$$

and, if s is taken such that

$$m-s+1+np=0, \text{ that is, } s=m+1+np, \quad (368)$$

(367) becomes

$$\frac{x^{m+1}(a+bx^n)^p + anp\int . x^m(a+bx^n)^{p-1}}{m+1+np}. \quad (369)$$

The value of the required integral is thus made to depend upon that of an integral, in which the exponent of the binomial $(a+bx^n)$ is diminished by unity; the value of this new integral may, by the same formula (369), be made to depend upon that of an integral, in which the exponent of the binomial is still farther diminished; and so on until *the exponent of the binomial is reduced to a fraction less than unity.*

Secondly. Instead of (360) let, now,

$$v = (a+bx^n)^s; \quad (370)$$

in which s is to be taken of any value, which may be found useful; whence

$$d_x v = nbs\,x^{n-1}(a+bx^n)^{s-1}, \quad (371)$$

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and if u is taken so as to satisfy (362),

$$u = \frac{1}{n\,b\,s} \cdot x^{m-n+1}(a + b\,x^n)^{p-s+1} \qquad (372)$$

$$d_x u = \frac{m-n+1}{n\,b\,s}\,x^{m-n}(a+b\,x^n)^{p-s+1} + \frac{p-s+1}{s}\,x^m\,(a+b\,x^n)^{p-s}$$

$$= \left(\frac{a(m-n+1)}{n\,b\,s}\,x^{m-n} + \frac{m+1+np-ns}{s}\,x^m\right)(a+b\,x^n)^{p-s}, \quad (373)$$

and if s is taken such that

$$m+1+np-ns = 0, \text{ that is, } s = \frac{m+1}{n} + p \qquad (374)$$

(373) becomes

$$d_x.u = \frac{a\,(m-n+1)}{b(m+1+np)}\,x^{m-n}\,(a+b\,x^n)^{p-s}, \qquad (375)$$

and, by (262), the integral of (359) is

$$\frac{x^{m-n+1}(a+b\,x^n)^{p+1} - a(m-n+1)\int . x^{m-n}(a+b\,x^n)^p}{b\,(m+1+np)}, \quad (376)$$

in which the exponent of the factor x^m of the binomial is diminished by that of x^n in the binomial, and *this exponent may by a repeated application of* (376) *be still farther diminished until it is less than n.*

Thirdly. By the successive use of (369 *and* 376), *the required integral may be made to depend upon one of a similar form, in which the exponent of the binomial* $(a + b\,x^n)$ *is less than unity, and that of its factor is less than n.*

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Fourthly. The development of $(a+b\,x^n)^p$ may be effected by the binomial theorem, either according to ascending or descending powers of x; it being better to use the ascending powers when

$$x^n < \frac{a}{b}, \tag{377}$$

and the descending powers when

$$x^n > \frac{a}{b}. \tag{378}$$

In one case the integral of (359) is

$$\int \cdot \left(a^p x^m + p\,a^{p-1} b\,x^{m+n} + \frac{p(p-1)}{1\,.\,2} a^{p-2} b^2\,x^{m+2n} + \&c.\right)$$

$$= \frac{a^p\,x^{m+1}}{m+1} + \frac{p\,a^{p-1}\,b\,x^{m+n+1}}{m+n+1} + \frac{p(p-1)a^{p-2}b^2 x^{m+2n+1}}{1\,.\,2\,.\,(m+2n+1)} + \&c.$$

$$= a^p\,x^{m+1}\left[\frac{1}{m+1} + \frac{p}{m+n+1}\cdot\frac{b\,x^n}{a}\right.$$

$$\left. + \frac{p\,(p-1)}{1.2.(m+2n+1)}\left(\frac{b\,x^n}{a}\right)^2 + \&c.\right] \tag{379}$$

and in the other case

$$\int \cdot \left(b^p\,x^{np+m} + p\,a\,b^{p-1}\,x^{np+m-n} + \frac{p(p-1)}{1\,.\,2} a^2 b^{p-2} x^{np+m-2n} + \&c.\right)$$

$$= \frac{b^p\,x^{np+m+1}}{np+m+1} + \frac{p\,a\,b^{p-1}\,x^{np+m-n+1}}{np+m-n+1} + \&c. \tag{380}$$

$$= b^p\,x^{np+m+1}\left(\frac{1}{np+m+1} + \frac{p}{np+m-n+1}\cdot\frac{a}{b\,x^n} + \&c.\right)$$

41. Examples.

1. Reduce the integral of $x^4(a+bx^3)^{\frac{3}{2}}$ to depend upon one, in which the exponent of $a+bx^3$ is less than unity, and the exponent of its factor is less than 3.

Solution. By putting in (369)

$$m=4, \quad n=3, \quad p=\tfrac{3}{2},$$

it gives

$$\int.x^4(a+bx^3)^{\frac{3}{2}} = \tfrac{2}{19}x^5(a+bx^3)^{\frac{3}{2}} + \frac{9a}{19}\int.x^4(a+bx^3)^{\frac{1}{2}},$$

and by putting in (376)

$$m=4, \quad n=3, \quad p=\tfrac{1}{2},$$

it gives

$$\int.x^4(a+bx^3)^{\frac{1}{2}} = \frac{2x^2(a+bx^3)^{\frac{3}{2}}}{13b} - \frac{4a}{13b}\int.x(a+bx^3)^{\frac{1}{2}},$$

so that, by substitution,

$$\int.x^4(a+bx^3)^{\frac{3}{2}} = \left(\frac{2}{19}x^5 + \frac{18}{247}\cdot\frac{ax^2}{b}\right)(a+bx^3)^{\frac{3}{2}}$$
$$-\frac{36}{247}\frac{a^2}{b}\int.x(a+bx^3)^{\frac{1}{2}}.$$

2. Develop the integral of $x(a+bx^3)^{\frac{1}{2}}$ according to powers of x.

Solution. By putting in (379 and 380)

$$m=1, \quad n=3, \quad p=\tfrac{1}{2},$$

$$\int . x (a + b x^3)^{\frac{1}{2}} = \sqrt{a} . x^2 \left[\tfrac{1}{2} + \frac{b x^3}{10 a} - \tfrac{1}{64} \left(\frac{b x^2}{a} \right)^2 + \&c. \right]$$

$$\text{or} \qquad = \sqrt{(b x^7)} \left[\tfrac{2}{7} + \frac{a}{b x^3} - \tfrac{1}{10} \left(\frac{a}{b x^3} \right)^2 - \&c. \right]$$

3. Reduce the integral of $x^4 (a^2 - x^2)^{\frac{4}{3}}$ to depend upon one, in which the exponent of $a^2 - x^2$ is less than unity, and that of its factor is less than 2.

$$\textit{Ans.} \; (\tfrac{3}{23} x^5 - \tfrac{24}{391} a^2 x^3 - \tfrac{216}{4301} a^4 x) (a^2 - x^2)^{\frac{4}{3}}$$

$$+ \tfrac{216}{4301} a^6 \int . (a^2 - x^2)^{\frac{1}{3}}.$$

4. Develop the integral of $(a^2 - x^2)^{\frac{1}{3}}$ according to powers of x.

$$\textit{Ans.} \quad a^{\frac{2}{3}} x \left(1 - \tfrac{1}{9} . \frac{x^2}{a^2} - \tfrac{1}{45} . \frac{x^4}{a^4} + \&c. \right)$$

$$\text{or} \qquad - x^{\frac{5}{3}} \left(\tfrac{3}{5} + \frac{a^2}{x^2} + \tfrac{1}{21} . \frac{a^4}{x^4} + \&c. \right)$$

5. Develop the integral of $x (1 + x^3)^{\frac{2}{3}}$ according to powers of x.

$$\textit{Ans.} \quad x^2 (\tfrac{1}{2} + \tfrac{2}{15} x^3 - \tfrac{1}{72} x^6 + \&c.)$$

$$\text{or} \qquad x^4 (\tfrac{1}{4} + \tfrac{2}{3} x^{-3} + \tfrac{1}{18} x^{-6} + \&c.)$$

42. *Problem. To integrate* (359) *for all real values of m, n, and p.*

Solution. First. When m is a negative integer and n a positive integer, the substituting of $m + n$ for m in (376),

freeing from fractions, dividing by $a\,(m+1)$, and transposing, give for the required integral

$$\frac{x^{m+1}\,(a+b\,x^n)^{p+1} - b\,(m+1+n\,p+n)\int. x^{m+n}\,(a+b\,x^n)^p}{a\,(m+1)}, \quad (381)$$

which formula, since m is negative, serves to increase the exponent of the factor of the binomial under the sign of integration, *until it becomes positive but less than n.*

Secondly. When p is negative, the substitution of $p+1$ for p in (369), gives by reduction for the value of the required integral

$$\frac{-x^{m+1}\,(a+b\,x^n)^{p+1} + (m+1+n\,p+n)\int. x^m\,(a+b\,x^n)^{p+1}}{a\,n\,(p+1)}, \quad (382)$$

which formula serves to increase the exponent of the binomial under the sign of integration, *until it becomes positive but less than unity.*

Thirdly. When m and n are fractions and n positive, let the common denominator of m and n be l, and let

$$x = y^l \quad (383)$$

whence

$$d_{c.y.}\,x = l\,y^{l-1} \quad (384)$$

and, by § 19, the intregal of (359) becomes

$$\int. l\,y^{ml+l-1}\,(a+b\,y^{ln})^p \quad (385)$$

in which the exponents of y are integers, so that it may be integrated by § 40, or the preceding part of this section.

Fourthly. When n is negative, a simple algebraic reduction gives

$$x^m (a+b\,x^n)^p = x^{m+np} (b+a\,x^{-n})^p, \tag{386}$$

the integration of which may be effected by § 40 or the preceding part of this section.

43. Examples.

1. Reduce the integral of $x^{-2}(1+x^3)^{-\frac{1}{3}}$ to depend upon one, in which the exponent of the binomial is positive and less than unity, and that of its factor is positive and less than 3.

Solution. The substitution of

$$m=-2,\ n=3,\ p=-\tfrac{1}{3},\ a=1,\ b=1,$$

in (381), gives

$$\textstyle\int x^{-2}(1+x^3)^{-\frac{1}{3}} = -x^{-1}(1+x^3)^{\frac{2}{3}} + \int x(1+x^3)^{-\frac{1}{3}},$$

the substitution of

$$m=1,\ n=3,\ p=-\tfrac{1}{3},\ a=1,\ b=1,$$

in (382), gives

$$\textstyle\int x(1+x^3)^{-\frac{1}{3}} = -\tfrac{1}{2}x^2(1+x^3)^{\frac{2}{3}} + 2\int x(1+x^3)^{\frac{2}{3}}.$$

Hence

$$\textstyle\int x^{-2}(1+x^3)^{-\frac{1}{3}} = -(x^{-1}+\tfrac{1}{2}x^2)(1+x^3)^{\frac{2}{3}} + 2\int x(1+x^3)^{\frac{2}{3}}$$

2. Reduce the integral of $x^{-3}(1+x^3)^{-\frac{2}{3}}$ to depend upon one, in which the exponent of the binomial is positive and less than unity, and that of its factor is positive and less than 3.

Ans. $\frac{1}{2}(x-x^{-2})(1+x^3)^{\frac{1}{3}} - \int (1+x^3)^{\frac{1}{3}}.$

3. Reduce the integral of $(1+x^5)^{-\frac{12}{5}}$ to depend upon one, in which the exponent of the binomial is positive and less than unity.

Ans. $\frac{1}{7}\, x\, (1+x^5)^{-\frac{7}{5}}\, (3+x^5-x^{10}) + \frac{4}{7} \int (1+x^5)^{\frac{3}{5}}$.

4. Reduce the integral of $x^{\frac{2}{3}}\,(a+b\,x^{\frac{4}{5}})^p$ to depend upon one of the same form, but in which the exponents are integral, except that of the binomial.

Solution. In (383) we have, for this case,

$$l = 15, \quad x = y^{15},$$

so that (385) gives

$$\int x^{\frac{2}{3}}\,(a+b\,x^{\frac{4}{5}})^p = 15 \int y^{24}\,(a+b\,y^{12})^p.$$

5. Reduce the integral of $x^{\frac{4}{5}}\,(a+b\,x^{\frac{2}{3}})^p$ to depend upon one of the same form, but in which the exponents are integral, except that of the biomial.

Ans. $15 \int y^{26}\,(a+b\,y^{10})^p$.

6. Reduce the integral of $x^{\frac{3}{4}}\,(a+b\,x^{-2})^{\frac{3}{8}}$ to depend upon one in which the exponent of x in the binomial is positive.

Ans. $\int (b+a\,x^2)^{\frac{3}{8}}$.

44. *Problem. To find the value of the definite integral*

$$\int_0^c x^m\,(a+b\,x^n)^p \tag{387}$$

in which

$$c = \sqrt[n]{-\frac{a}{b}} \tag{388}$$

and m, n, and p are positive.

Value of binomial definite integral.

Solution. The substitution of

$$a = -b\,c^n \tag{389}$$

reduces (387) to

$$b^p \int_0^c x^m\,(x^n - c^n)^p = (-b)^p \int_0^c x^m\,(c^n - x^n)^p. \tag{390}$$

First. The term of (369)

$$x^{m+1}\,(a + b\,x^n)^p = b^p\,x^{m+1}\,(x^n - c^n)^p \tag{391}$$

is zero when $x = 0$, and when $x = c$. (392)

Hence (369) gives for the value of (387)

$$\frac{a\,n\,p}{m+1+n\,p} \cdot \int_0^c \cdot\, x^m\,(a + b\,x^n)^{p-1}, \tag{393}$$

and, in the same way, by changing p to $p-1$, $p-2$, &c., (369) gives

$$\int_0^c x^m\,(a+b\,x^n)^{p-1} = \frac{a\,n\,(p-1)}{m+1+n(p-1)} \int_0^c x^m (a+b\,x^n)^{p-2} \tag{394}$$

$$\int_0^c x^m\,(a+b\,x^n)^{p-2} = \frac{a\,n\,(p-2)}{m+1+n(p-2)} \int_0^c x^m (a+b\,x^n)^{p-3} \tag{395}$$

&c.

The substitution of each successive value, in the preceding one, gives for the value of (387), if p_0 is the greatest integer in p,

$$\frac{(a\,n)^{p_0}\,p\,(p-1)\,(p-2)\,\ldots\,(p-p_0+1)}{(m+1+n\,p)\,[m+1+n(p-1)]\ldots[m+1+n(p-p_0+1)]} \times \int_0^c x^m\,(a + b\,x^n)^{p-p_0} \tag{396}$$

Value of binomial definite integral.

Secondly. In the same way, (376) gives

$$\int_0^c . x^m (a + b x^n)^p = -\frac{a(m-n+1)}{b(m+1+np)} \int_0^c . x^{m-n} (a+bx^n)^p \quad (397)$$

$$\int_0^c . x^{m-n} (a + b x^n)^p = -\frac{a(m-2n+1)}{b[m+1+n(p-1)]} \int_0^c x^{m-2n} (a+b x^n)^p \quad (398)$$

&c.

and for the final value of (387), if h is the greatest integral number of times, which n is contained in m,

$$c^{nh} \frac{(m-n+1) \ldots\ldots (m-hn+1)}{(m+1+np) \ldots\ldots [m+1+n(p-h+1)]} \int_0^c x^{m-hn} (a+b x^n)^p \quad (399)$$

Thirdly. The series (379) gives for the value of (387)

$$a^p c^{m+1} \left(\frac{1}{m+1} - \frac{p}{m+n+1} + \frac{p(p-1)}{1.2(m+2n+1)} - \text{\&c.} \right) \quad (400)$$

45. *Corollary.* In the particular case, in which

$$m = 0, \quad n = 2, \quad p = -\tfrac{1}{2}, \quad (401)$$

we have

$$a = -b c^2, \quad c = \sqrt{-\frac{a}{b}}, \quad (402)$$

and by (331)

$$\int . \frac{1}{\sqrt{(a+b x^2)}} = -\frac{1}{\sqrt{-b}} \cos.^{[-1]} \frac{b x}{\sqrt{(-ab)}}; \quad (403)$$

whence

$$\int_0^c . \frac{1}{\sqrt{(a+b x^2)}} = -\frac{1}{\sqrt{-b}} \cos.^{[-1]} 1 + \frac{1}{\sqrt{-b}} \cos.^{[-1]} 0$$

$$= \frac{1}{\sqrt{-b}} . \frac{\pi}{2} \quad (404)$$

Value of binomial definite integral.

46. *Corollary.* It follows from (404), that if

$$b = -g^2, \quad c = \frac{\sqrt{a}}{g} \tag{405}$$

$$\int_0^c \frac{1}{\sqrt{(a-g^2x^2)}} = \frac{1}{g} \cdot \frac{\pi}{2}. \tag{406}$$

47. Examples.

1. Find the value of the definite integral $\int_0^c \frac{x^4}{\sqrt{(a-g^2x^2)}}$, in which $c = \frac{\sqrt{a}}{g}$.

Solution. The substitution of

$$m = 4, \quad n = 2, \quad p = -\tfrac{1}{2}, \quad h = 2$$

in (399) gives, by (406),

$$\int_0^c \frac{x^4}{\sqrt{(a-g^2x^2)}} = \frac{a^2}{g^4} \cdot \frac{3.1}{4.2} \int_0^c \frac{1}{\sqrt{(a-g^2x^2)}} = \frac{3a^2}{8g^5} \cdot \frac{\pi}{2}.$$

2. Find the value of the definite integral $\int_0^a \frac{x^6}{\sqrt{(a^2-x^2)}}$.

Ans. $\frac{1.3.5}{2.4.6} \cdot \frac{\pi a^6}{2}$.

3. Find the value of the definite integral $\int_0^a \frac{x^3}{\sqrt{(a^2-x^2)}}$.

Solution. Equation (399) gives

$$\int_0^a \frac{x^3}{\sqrt{(a^2-x^2)}} = \frac{2a^2}{3} \int_0^a \frac{x}{\sqrt{(a^2-x^2)}}.$$

Value of binomial definite integral.

But by (324)

$$\int \cdot \frac{x}{\sqrt{(a^2 - x^2)}} = -\sqrt{(a^2 - x^2)},$$

whence
$$\int_0^a \cdot \frac{x}{\sqrt{(a^2 - x^2)}} = a,$$

and
$$\int_0^a \cdot \frac{x^3}{\sqrt{(a^2 - x^2)}} = \tfrac{2}{3}\, a^3.$$

4. Find the value of the definite integral $\int_0^a \cdot \frac{x^5}{\sqrt{(a^2 - x^2)}}$.

Ans. $\frac{2\,.\,4}{3\,.\,5}\, a^5.$

5. Find the value of the definite integral $\int_0^c \cdot \sqrt{(a + b\,x^2)}$ where $c = \sqrt{-\frac{a}{b}}$.

Solution. The substitution of

$$m = 0, \quad n = 2, \quad p = \tfrac{1}{2},$$

in (393) gives by (404)

$$\int_0^c \cdot \sqrt{(a + b\,x^2)} = \frac{a}{2} \int_0^\infty \cdot \frac{1}{\sqrt{(a + b\,x^2)}} = \frac{\pi\, a}{4\sqrt{-b}}. \quad (407)$$

6. Find the value of the definite integral $\int_0^a \cdot \sqrt{(a^2 - x^2)}$.

Ans. $\frac{\pi\, a^2}{4}.$

7. Find the value of the definite integral $\int_0^a \cdot x^2 \sqrt{(a^2 - x^2)}$.

Ans. $\tfrac{1}{4} \cdot \frac{\pi\, a^4}{4}.$

8. Find the value of the definite integral $\int_0^a \cdot x \sqrt{(a^2 - x^2)}$.

Ans. $\tfrac{1}{3}\, a^3.$

9. Find the value of the definite integral $\int_0^a . x^3 \sqrt{(a^2 - x^2)}$.

Ans. $\frac{2}{5} \cdot \frac{1}{3} a^5$.

10. Find the value of the definite integral $\int_0^a . (a^2 - x^2)^{\frac{5}{2}}$.

Ans. $\frac{5}{32} \pi a^6$.

11. Find the value of the definite integral $\int_0^a . x^2 (a^2 - x^2)^{\frac{3}{2}}$.

Ans. $\frac{1}{6} \cdot \frac{3}{16} \pi a^6$.

12. Find the value of the definite integral $\int_0^a . x^3 (a^2 - x^2)^{\frac{3}{2}}$.

Ans. $\frac{2}{7} \cdot \frac{1}{5} a^7$.

13. Find the value of the definite integral $\int_0^a . x (a^2 - x^2)^{\frac{5}{2}}$.

Ans. $\frac{1}{7} a^7$.

CHAPTER IV.

INTEGRATION OF LOGARITHMIC FUNCTIONS.

48. *Problem. To integrate*

$$f.\,x.\,(\log.\,F.\,x)^n \tag{408}$$

Solution. First. When n is positive, the substitution of

$$u = (\log.\,F.\,x)^n,\ d_c.\,v = f.\,x, \tag{409}$$

in (262) gives for the integral of (408)

$$(\log.\,F.\,x)^n \textstyle\int.\,f.\,x - n\displaystyle\int.\frac{(\log.\,F.\,x)^{n-1}\,d_c.\,F.\,x.\textstyle\int.f.\,x}{F.\,x} \tag{410}$$

by which formula the exponent of the logarithm is diminished by unity, and may be still farther reduced by the repeated application of the same formula.

Secondly. When n is negative and differs from -1, the substitution of

$$v = (\log.\,F.\,x)^{n+1},\ u = \frac{f.\,x.\,F.\,x}{(n+1)\,d_c.\,F.\,x}, \tag{411}$$

in (262) gives for the integral of (408)

$$\frac{f.x.F.x.(\log.F.x)^{n+1}}{(n+1)\,d_c.\,F.\,x} - \frac{1}{n+1}\int.(\log.F.x)^{n+1}.d_c.\frac{f.x.F.x}{d_c.F.x} \tag{412}$$

by which the exponent of the logarithm is increased by unity.

9*

Integration of logarithmic functions.

Thirdly. When $$n = -1 \tag{413}$$

the only useful reduction occurs in the case of

$$f.\,x = \frac{d_{c}.\,F.\,x}{F.\,x} = d_{c}.\,\log.\,F.\,x \tag{414}$$

in this case, since $$d_{c}.\,\log.^2\,F.\,x = \frac{d_{c}.\,F.\,x}{F.\,x\,\log.\,F.\,x}, \tag{415}$$

the required integral is $$\int.\frac{d_{c}.\,F.\,x}{F.\,x\,\log.\,F.\,x} = \log.^2\,F.\,x. \tag{416}$$

Fourthly. The particular case of (414), gives also a different solution of the general problem; for in this case the integral of (408) is

$$f.\,(\log.\,F.\,x)^n.d^c.\,\log.\,F.\,x = \frac{(\log.\,F.\,x)^{n+1}}{n+1}. \tag{417}$$

In other cases, the integration can only be advanced in the form of a series.

49. Examples.

1. Integrate $x^m\,(\log.\,x)^2$.

Solution. First. When m differs from -1, in which case (410) gives

$$\int.\,x^m\,(\log.\,x.)^2 = \frac{x^{m+1}(\log.\,x)^2}{m+1} - \frac{2}{m+1}\int.\,x^m\,\log.\,x$$

$$\int.\,x^m\,(\log.\,x) = \frac{x^{m+1}\,\log.\,x}{m+1} - \frac{1}{m+1}\int.\,x^m$$

$$= \frac{x^{m+1}\,\log.\,x}{m+1} - \frac{x^{m+1}}{(m+1)^2}$$

so that the required integral is

$$\frac{x^{m+1}}{m+1}\left[(\log.\, x)^2 - \frac{2}{m+1}\log.\, x + \frac{2}{(m+1)^2}\right].$$

Secondly. When $m = -1$

(417) gives for the required integral

$$\tfrac{1}{3}(\log.\, x)^3.$$

2. Integrate $x^m (\log.\, x)^3$.

Ans. When m differs from -1, it is

$$\frac{x^{m+1}}{m+1}\left[(\log.\, x)^3 - \frac{3(\log.\, x)^2}{m+1} + \frac{3.\,2.\,\log.\, x}{(m+1)^2} - \frac{3\,.\,2\,.\,1}{(m+1)^3}\right]$$

and when $m = -1$

it is $\tfrac{1}{4}(\log.\, x)^4$.

3. Integrate $f.\, x.\, \log.\, x$.

Ans. When $f.\, x$ differs from x^{-1}, it is

$$\log.\, x \int . f.\, x - \int . \frac{\int . f.\, x}{x}$$

and when $f.\, x = \frac{1}{x}$

it is $\tfrac{1}{2}(\log.\, x)^2$.

4. Integrate $\frac{(\log.\, x)^n}{x}$ when n differs from -1.

$$Ans.\ \frac{(\log.\, x)^{n+1}}{n+1}. \qquad (418)$$

5. Integrate $\frac{\log. x}{(1-x)^2}$.

$$Ans.\ \frac{x \log. x}{1-x} + \log. (1-x).$$

6. Integrate $\frac{1}{x \log. x}$. *Ans.* $\log.^2 x$.

50. *Problem.* To find the value of the definite integral

$$\int_0^1 . (-\log. x)^{\frac{n}{2}} \tag{419}$$

in which n is an integer greater than -2.

Solution. First. In this case, (408 - 410) give

$$f. x = 1, \quad F. x = x, \quad \int . f. x = x; \tag{420}$$

$$\int . (-\log. x.)^{\frac{n}{2}} = x(-\log. x)^{\frac{n}{2}} + \frac{n}{2} \int . (-\log. x)^{\frac{n}{2}-1} \tag{421}$$

in which when $x = 0$, or $= 1$, (422)

the first term of the second member vanishes as in example 2 of B. II. § 109, so that the required integral becomes

$$\frac{n}{2} \int_0^1 .(-\log. x)^{\frac{n}{2}-1}. \tag{423}$$

By this process, then, the exponent of $(-\log. x)$ is diminished by unity; and a continued repetition of it gives for the value of (419)

$$\frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right) \ldots\ldots \left(\frac{n}{2}-h+1\right) \int_0^1 .(-\log. x)^{\frac{n}{2}-h} \tag{424}$$

in which h is an integer not greater than $\frac{n}{2}+1$.

Logarithmic definite integrals.

Secondly. When n is even, let

$$h = \tfrac{1}{2} n \tag{425}$$

and (424) gives

$$\textstyle\int_0^1 \cdot (-\log. x)^h = 1 . 2 . 3 \ldots\ldots h. \tag{426}$$

Thirdly. When n is odd, and positive, let

$$h = \tfrac{1}{2} n + \tfrac{1}{2}, \tag{427}$$

and (424) gives

$$\textstyle\int_0^1 . (-\log. x)^{h-\frac{1}{2}} = (h-\tfrac{1}{2})(h-\tfrac{3}{2}) \ldots \tfrac{3}{2} . \tfrac{1}{2} \int_0^1 . (-\log. x)^{-\frac{1}{2}}. \tag{428}$$

Fourthly. When $\quad n = -1$

let $\quad K = \int_0^1 . (-\log. x)^{-\frac{1}{2}}. \quad$ (429)

The substitution of

$$x = a^{y^2}, \tag{430}$$

in which a is supposed less than unity, so that $(-\log. a)$ is positive, gives

$$-\log. x = -y^2 \log. a, \quad d_{c.y}. x = 2 y a^{y^2}. \log. a; \tag{431}$$

and when $\quad x = 0, \; y = \infty,$

$$x = 1, \; y = 0; \tag{432}$$

$$K = -2 \int_\infty^0 . a^{y^2} (-\log. a)^{\frac{1}{2}}, \tag{433}$$

and $\quad K (-\log. a)^{-\frac{1}{2}} = -2 \int_\infty^0 . a^{y^2}. \quad$ (434)

But, by taking the integrals relatively to a, we have

$$\int_0^1 . (-\log. a)^{-\frac{1}{2}} = K, \tag{435}$$

Logarithmic definite integrals.

$$\int_0^1 . a^{y^2} = \frac{1}{y^2+1} a^{y^2+1} = \frac{1}{y^2+1}; \qquad (436)$$

and, therefore, the integral of (434) with reference to a, is by (304)

$$K^2 = -2 \int_\infty^0 . \frac{1}{y^2+1} = -2 (\tan.^{[-1]} 0 - \tan.^{[-1]} \infty)$$
$$= -2 (0 - \tfrac{1}{2} \pi) = \pi. \qquad (437)$$

or $$K = \int_0^1 . (-\log. x)^{-\frac{1}{2}} = \int_0^1 . \frac{1}{\sqrt{(\log. \frac{1}{x})}} = \sqrt{\pi}. \qquad (438)$$

51. *Corollary.* The substitution of (438) in (428) gives

$$\int_0^1 . (\log. \tfrac{1}{x})^{h-\frac{1}{2}} = \frac{1.3.5.....(2h+1)}{2^h} \sqrt{\pi}. \qquad (439)$$

52. *Corollary.* The substitution of

$$x = y^{m+1}, \quad \text{or} \quad \log. x = (m+1) \log. y, \qquad (440)$$

whence $$d_{c.y}. x = (m+1) y^m \qquad (441)$$

in (426 and 439) gives; by dividing, in one case, by $(m+1)^{h+1}$; and, in the other, by $(m+1)^{h+\frac{1}{2}}$

$$\int_0^1 . y^m (\log. \tfrac{1}{y})^h = \frac{1.2.3.....h}{(m+1)^{h+1}}; \qquad (442)$$

$$\int_0^1 . y^m (\log. \tfrac{1}{y})^{h-\frac{1}{2}} = \frac{1.3.5....(2h+1)}{2^h (m+1)^{h+\frac{1}{2}}} \sqrt{\pi}. \qquad (443)$$

53. ***Problem.*** ***To integrate***

$$F. (e^{f.x}) . (f. x)^n d_c. f. x. \qquad (444)$$

Solution. Let
$$y = e^{f.x}, \tag{445}$$
whence
$$d_{c.y}.\,x = (e^{f.x}\,d_c.f.x)^{-1} = (y\,d_c.f.x)^{-1} \tag{446}$$
$$\log.\,y = f.\,x, \tag{447}$$
and the integral of (444) is, by § 19,
$$\int.\,(\log.\,y)^n.\,\frac{F.\,y}{y}, \tag{448}$$
which may be found by § 48.

54. *Corollary.* When
$$n = 0 \tag{449}$$
(448) gives
$$\int.\,F.\,(e^{f.x}).\,d_c.f.\,x = \int.\,\frac{F.\,y}{y}. \tag{450}$$

55. Examples.

1. Integrate $e^{ax}\sqrt{(1-e^{2ax})}$.

In this case if
$$f.\,x = a\,x,\quad d_c f.\,x = a$$
$$F.\,y = \tfrac{1}{a}\,y\sqrt{(1-y^2)}$$
(450) gives
$$\begin{aligned}\int.\,e^{ax}\sqrt{(1-e^{2ax})} &= \tfrac{1}{a}\int.\sqrt{(1-y^2)}\\ &= \tfrac{1}{2a}\,y\sqrt{(1-y^2)} + \tfrac{1}{2}\sin.^{[-1]}y\\ &= \tfrac{1}{2a}e^{ax}\sqrt{(1-e^{2ax})} + \tfrac{1}{2}\sin.^{[-1]}e^{ax}.\end{aligned}$$

2. Integrate e^{ax}. *Ans.* $\frac{1}{a}\,e^{ax}$. (451)

3. Integrate $x\,e^{ax}$. *Ans.* $\left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}$.

Potential integrals.

4. Integrate a^x. *Ans.* $\frac{a^x}{\log. a}$.

5. Integrate $\text{Sin.}(kx + a)$. *Ans.* $\frac{1}{k}\text{Cos.}(kx+a)$. (452)

6. Integrate $\text{Cos.}(kx + a)$. *Ans.* $\frac{1}{k}\text{Sin.}(kx+a)$. (453)

56. *Problem. To integrate*

$$f.(\text{Sin.}\, kx, \quad \text{Cos.}\, kx). \tag{454}$$

Solution. Let

$$y = \text{Sin.}\, kx, \quad \text{or} \quad kx = \text{Sin.}^{[-1]}\, y; \tag{455}$$

and by (127 and 143)

$$\text{Cos.}\, kx = \sqrt{(1+y^2)}, \quad k\, d_{c.y}.\, x = \frac{1}{\sqrt{(1+y^2)}}; \tag{456}$$

whence the integral of (454) is, by § 19,

$$\int. \tfrac{1}{k} f.[y, \sqrt{(1+y^2)}].(1+y^2)^{-\frac{1}{2}}, \tag{457}$$

which can be found by § 31.

57. Examples.

1. Integrate $\text{Sin.}^m\, kx\,.\, \text{Cos.}\, kx$.

Solution. In this case, (457) becomes

$$\int. y^m = \frac{y^{m+1}}{m+1} = \frac{\text{Sin.}^{m+1}\, kx}{(m+1)\, k}. \tag{458}$$

2. Integrate $\text{Cos.}^m\, kx.\, \text{Sin.}\, kx$. *Ans.* $\frac{\text{Cos.}^{m+1}\, kx}{(m+1)\, k}$.

3. Integrate $\text{Tang.}\, kx$. *Ans.* $\frac{1}{k}\log.\text{Cos.}\, kx$. (459)

Potential integrals.

4. Integrate Cotan. kx. *Ans.* $\frac{1}{k}$ log. Sin. kx. (460)

5. Integrate Sec. kx. *Ans.* $\frac{1}{k}$ tan.$^{[-1]}$ Sin. kx. (461)

6. Integrate Cosec. kx. *Ans.* $\frac{1}{2k}$ log. $\left(\frac{\text{Cos.}\,kx-1}{\text{Cos.}\,kx+1}\right)$,

or, by (140), $\frac{1}{k}$ log. Tan. $\frac{1}{2}kx$. (462)

7. Integrate e^{ax} Sin. kx. (463)

Solution. Since by (121 and 122)

$$\text{Sin.}\,kx = \tfrac{1}{2}\,(e^{kx} - e^{-kx}),$$

$$\text{Cos.}\,kx = \tfrac{1}{2}\,(e^{kx} + e^{-kx})\,;$$

we have $$e^{ax}\,\text{Sin.}\,kx = \tfrac{1}{2}\left(e^{(a+k)x} - e^{(a-k)x}\right),$$

$$\int . e^{ax}\,\text{Sin.}\,kx = \frac{e^{(a+k)x}}{2(a+k)} - \frac{e^{(a-k)x}}{2(a-k)}$$

$$= e^{ax}\left(\frac{a(e^{kx}-e^{-kx}) - k(e^{kx}+e^{-kx})}{2\,(a^2-k^2)}\right)$$

$$= e^{ax}\left(\frac{a\,\text{Sin.}\,kx - k\,\text{Cos.}\,kx}{a^2-k^2}\right). \quad (464)$$

8. Integrate e^{ax} Cos. kx.

Ans. $e^{ax}\left(\frac{a\,\text{Cos.}\,kx - k\,\text{Sin.}\,kx}{a^2-k^2}\right)$. (465)

9. Integrate e^{ax} Sin. ax. *Ans.* $\frac{1}{4a}e^{2ax} - \frac{1}{2}x$. (466)

10. Integrate e^{ax} Cos. ax. *Ans.* $\frac{1}{4a}e^{2ax} + \frac{1}{2}x$. (467)

11. Integrate $e^{a\,\text{Sin.}\,kx}$ Cos. kx. *Ans.* $\frac{1}{ak}e^{a\,\text{Sin.}\,kx}$. (468)

12. Integrate $e^{a\,\text{Cos.}\,kx}$ Sin. kx. *Ans.* $\frac{1}{ak}e^{a\,\text{Cos.}\,kx}$. (469)

Potential integrals.

58. *Corollary.* The differential coefficients of (451, 452, 453, 458, 464, 465) with respect to a, k, or m, give

$$\int . d_{c.a} e^{ax} = d_{c.a} . \tfrac{1}{a} e^{ax},$$

$$\int . x e^{ax} = d_{c.a} . \tfrac{1}{a} e^{ax}$$

$$= \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}, \qquad (470)$$

$$\int . x^2 e^{ax} = d^2_{c.a} . \tfrac{1}{a} e^{ax}$$

$$= \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax}, \qquad (471)$$

$$\int . x^n e^{ax} = d^n_{c.a} . \tfrac{1}{a} e^{ax}; \qquad (472)$$

$$\int . x \operatorname{Cos.} (kx + a) = d_{c.k} . \tfrac{1}{k} \operatorname{Cos.} (kx + a) \qquad (473)$$

$$= \frac{x}{k} \operatorname{Sin.} (kx + a) - \frac{1}{k^2} \operatorname{Cos.} (kx + a)$$

$$\int . x^2 \operatorname{Sin.} (kx + a) = d^2_{c.k} . \tfrac{1}{k} \operatorname{Cos.} (kx + a) \qquad (474)$$

$$= \left(\frac{x^2}{k} + \frac{2}{k^3}\right) \operatorname{Cos.} (kx + a) - \frac{2x}{k^2} \operatorname{Sin.} (kx + a),$$

$$\int . x^{2n+1} \operatorname{Cos.} (kx + a) = d^{2n+1}_{c.k} . \tfrac{1}{k} \operatorname{Cos.} (kx + a), \qquad (475)$$

$$\int . x^{2n} \operatorname{Sin.} (kx + a) = d^{2n}_{c.k} . \tfrac{1}{k} \operatorname{Cos.} (kx + a), \qquad (476)$$

$$\int . x^{2n+1} \operatorname{Sin.} (kx + a) = d_{c.a} . d^{2n+1}_{c.k} . \tfrac{1}{k} \operatorname{Cos.} (kx + a)$$

$$= d^{2n+1}_{c.k} . \tfrac{1}{k} \operatorname{Sin.} (kx + a), \qquad (477)$$

$$\int . x^{2n} \operatorname{Cos.} (kx + a) = d^{2n}_{c.k} . \tfrac{1}{k} \operatorname{Sin.} (kx + a), \qquad (478)$$

$$\int . \operatorname{Sin.}^m kx . \operatorname{Cos.} kx . \operatorname{log.} \operatorname{Sin.} kx = d_{c.m.} \frac{\operatorname{Sin.}^{m+1} kx}{(m+1)k} \qquad (479)$$

$$= \left(\operatorname{log.} \operatorname{Sin.} kx - \frac{1}{m+1}\right) \frac{\operatorname{Sin.}^{m+1} kx}{(m+1)k},$$

Potential integrals.

$$\int \mathrm{Cos.}^m kx \,.\, \mathrm{Sin.}\, kx \,.\, \log.\, \mathrm{Cos.}\, kx = d_{c.m} \,.\, \frac{\mathrm{Cos.}^{m+1} kx}{(m+1)k} \tag{480}$$

$$= \left(\log.\, \mathrm{Cos.}\, kx - \frac{1}{m+1}\right) \frac{\mathrm{Cos.}^{m+1} kx}{(m+1)k}$$

$$\int x\, e^{ax}\, \mathrm{Sin.}\, kx = d_{c.a} \left(e^{ax} . \frac{a\, \mathrm{Sin.}\, kx - k\, \mathrm{Cos.}\, kx}{a^2 - k^2}\right) \tag{481}$$

$$= \left(x - \frac{2a}{a^2 - k^2}\right) e^{ax} . \frac{a\, \mathrm{Sin.}\, kx - k\, \mathrm{Cos.}\, kx}{a^2 - k^2} + \frac{e^{ax}\, \mathrm{Sin.}\, kx}{a^2 - k^2} \tag{482}$$

$$\int x^n e^{ax}\, \mathrm{Sin.}\, kx = d^n_{c.a} \left(e^{ax} . \frac{a\, \mathrm{Sin.}\, kx - k\, \mathrm{Cos.}\, kx}{a^2 - k^2}\right). \tag{483}$$

59. *Corollary.* Equations (121 and 123) give

$$x^n e^{ax}\, \mathrm{Sin.}\, ax = \tfrac{1}{2} x^n e^{2ax} - \tfrac{1}{2} x^n, \tag{484}$$

$$x^n e^{ax}\, \mathrm{Cos.}\, ax = \tfrac{1}{2} x^n e^{2ax} + \tfrac{1}{2} x^n; \tag{485}$$

so that by (472)

$$\int x^n e^{ax}\, \mathrm{Sin.}\, ax = \tfrac{1}{2} d^n_{c.2a} . \frac{1}{2a} e^{2ax} - \frac{1}{2(n+1)} x^{n+1} \tag{486}$$

$$= \frac{1}{2^{n+1}} d^n_{c.a} . \frac{1}{2a} e^{2ax} - \frac{1}{2(n+1)} x^{n+1},$$

$$\int x^n e^{ax}\, \mathrm{Cos.}\, ax = \tfrac{1}{2} d_{c.2a} . \frac{1}{2a} e^{2ax} + \frac{1}{2(n+1)} x^{n+1} \tag{487}$$

$$= \frac{1}{2^{n+1}} d^n_{c.a} . \frac{1}{2a} e^{2ax} + \frac{1}{2(n+1)} x^{n+1}.$$

60. *Corollary.* When

$$n = 1$$

(486 and 487) become

$$\int . x e^{az} \text{Sin.} a x = \tfrac{1}{4} d_{c.a} . \tfrac{1}{2a} e^{2az} - \tfrac{1}{4} x^2 \qquad (488)$$

$$= \tfrac{1}{4} e^{2ax} \left(\frac{x}{a} - \frac{1}{2a^2} \right) - \tfrac{1}{4} x^2,$$

$$\int . x e^{az} \text{Cos.} a x = \tfrac{1}{4} d_{c.a} . \tfrac{1}{2a} e^{2az} + \tfrac{1}{4} x^2 \qquad (489)$$

$$= \tfrac{1}{4} e^{2ax} \left(\frac{x}{a} - \frac{1}{2a^2} \right) + \tfrac{1}{4} x^2.$$

61. *Problem.* To find the value of the definite integral

$$\int_0^\infty . x^n e^{-ax^2}, \qquad (490)$$

in which a is positive and n is zero or a positive integer.

Solution. Let

$$e^{-x^2} = y, \quad e^{x^2} = \tfrac{1}{y}, \qquad (491)$$

$$x^2 = -\log. y = \log. \tfrac{1}{y}, \qquad (492)$$

$$2 x d_{c.y} . x = -\tfrac{1}{y}, \quad d_{c.y} . x = -\frac{1}{2xy}; \qquad (493)$$

so that when $x = 0, \quad y = 1,$ (494)

$$x = \infty, \quad y = 0. \qquad (495)$$

The value of (490) is, by § 19,

$$\tfrac{1}{2} \int_0^1 . (\log. \tfrac{1}{y})^{\frac{n-1}{2}} y^{a-1}. \qquad (496)$$

Exponential definite integrals.

Hence, by (442 and 443), when n is odd,

$$\int_0^\infty . x^n e^{-ax^2} = \frac{1.2.3.\ldots.(\frac{1}{2}n - \frac{1}{2})}{2a^{\frac{1}{2}(n+1)}}, \tag{497}$$

and when n is even

$$\int_0^\infty . x^n e^{-ax^2} = \frac{1.3.5.\ldots.(n+1)}{2(2a)^{\frac{1}{2}n}} \sqrt{\frac{\pi}{a}}. \tag{498}$$

62. *Corollary.* When

$$n = 0, \quad a = 1,$$

(498) gives

$$\int_0^\infty . e^{-x^2} = \tfrac{1}{2}\sqrt{\pi}. \tag{499}$$

63. *Corollary.* By reversing the sign of x (497 – 499) give, when n is odd,

$$\int_{-\infty}^0 . x^n e^{-ax^2} = -\frac{1.2.3.\ldots.(\frac{1}{2}n - \frac{1}{2})}{2a^{\frac{1}{2}(n+1)}}, \tag{500}$$

when n is even

$$\int_{-\infty}^0 . x^n e^{-ax^2} = \frac{1.3.5.\ldots.(n+1)}{2(2a)^{\frac{1}{2}n}} \sqrt{\frac{\pi}{a}}, \tag{501}$$

$$\int_{-\infty}^0 . e^{-x^2} = \tfrac{1}{2}\sqrt{\pi}. \tag{502}$$

64. *Corollary.* The sums of (497 and 500), of (498 and 501), of (499 and 502), give, when n is odd,

$$\int_{-\infty}^\infty . x^n e^{-ax^2} = 0, \tag{503}$$

when n is even,

$$\int_{-\infty}^{\infty} . x^n e^{-ax^2} = \frac{1.3.5 \ldots\ldots (n+1)}{(2a)^{\frac{1}{2}n}} \sqrt{\frac{\pi}{a}}, \tag{504}$$

$$\int_{-\infty}^{\infty} . e^{-x^2} = \sqrt{\pi}. \tag{505}$$

65. *Corollary*. The substitution of

$$x + \frac{b}{2a} \tag{506}$$

for x in (504) gives by § 18, when $n = 0$,

$$\int_{-\infty}^{\infty} . e^{-\left(ax^2 + bx + \frac{b^2}{4a}\right)} = \sqrt{\frac{\pi}{a}}; \tag{507}$$

which, multiplied by $e^{-c + \frac{b^2}{4a}}$, gives

$$\int_{-\infty}^{\infty} . e^{-(ax^2 + bx + c)} = e^{\frac{b^2}{4a} - c} \sqrt{\frac{\pi}{a}}. \tag{508}$$

66. *Corollary*. The differential coefficients of (508), with reference to a and b, are

$$\int_{-\infty}^{\infty} . x^2 e^{-(ax^2 + bx + c)} = \left(\frac{b^2}{4a^2} + \frac{1}{2a}\right) e^{\frac{b^2}{4a} - c} \sqrt{\frac{\pi}{a}}, \tag{509}$$

$$\int_{-\infty}^{\infty} . x e^{-(ax^2 + bx + c)} = -\frac{b}{2a} e^{\frac{b^2}{4a} - c} \sqrt{\frac{\pi}{a}}. \tag{510}$$

67. Examples.

1. Find the value of the definite integral

$$\int_{-\infty}^{\infty} . x e^{-(ax^2 + c)} \operatorname{Sin.} kx.$$

Ans. $\frac{k}{2a} e^{\frac{k^2}{4a} - c} \sqrt{\frac{\pi}{a}}$.

Potential definite integrals.

2. Find the value of the definite integral

$$\int_{-\infty}^{\infty} . \, x \, e^{-(a x^2 + c)} \, \text{Cos.} \, k x. \qquad \textit{Ans.} \quad 0.$$

3. Find the value of the definite integral

$$\int_{-\infty}^{\infty} . \, (m x^2 + n) \, e^{-(a x^2 + c)} \, \text{Sin.} \, k x. \qquad \textit{Ans.} \quad 0.$$

4. Find the value of the definite integral

$$\int_{-\infty}^{\infty} . \, (m x^2 + n) \, e^{-(a x^2 + c)} \, \text{Cos.} \, k x.$$

$$\textit{Ans.} \quad \left(\frac{m k^2}{4 a^2} + \frac{m}{2 a} + n \right) e^{\frac{k^2}{4 a} - c} \sqrt{\frac{\pi}{a}}.$$

5. Find the value of the definite integral

$$\int_0^{\infty} . \, e^{-a x},$$

in which a is positive. *Ans.* $\frac{1}{a}$. (511)

6. Find the value of the definite integral

$$\int_0^{\infty} . \, x^n \, e^{-a x},$$

in which a is positive.

$$\textit{Ans.} \quad \frac{1 . 2 \ldots . n}{a^{n+1}}. \qquad (512)$$

CHAPTER V.

INTEGRATION OF CIRCULAR FUNCTIONS.

68. *Problem.* *To integrate*

$$f.\,(\sin.\,k\,x,\ \cos.\,k\,x). \tag{513}$$

Solution. The substitution of

$$x = y\sqrt{-1},\quad d_{c.y}.\,x = \sqrt{-1}, \tag{514}$$

gives by (121 and 122)

$$\sin.\,k\,x = \sqrt{-1}\,.\,\mathrm{Sin.}\,k\,y, \tag{515}$$

$$\cos.\,k\,x = \mathrm{Cos.}\,k\,y; \tag{516}$$

and the integral of (513) is

$$\textstyle\int.\sqrt{-1}\,.\,f.\,(\sqrt{-1}\,.\,\mathrm{Sin.}\,k\,y,\ \mathrm{Cos.}\,k\,y), \tag{517}$$

which may be found by § 56.

69. Examples.

1. Integrate $\sin.^m k\,x\,.\,\cos.\,k\,x$.

Solution. In this case (517) becomes

$$(-1)^{\frac{1}{2}(m+1)}\,\mathrm{Sin.}^m\,k\,y\,.\,\mathrm{Cos.}\,k\,y\,;$$

whence by (458, 121, 122, and 514)

$$\int . \sin.^m k x . \cos. k x = (-1)^{\frac{1}{2}(m+1)} \frac{\text{Sin.}^{m+1} k y}{(m+1) k}$$

$$= \frac{\sin.^{m+1} k x}{(m+1) k}. \qquad (518)$$

2. Integrate $\cos.^m k x . \sin. k x$.

Ans. $-\dfrac{\cos.^{m+1} k x}{(m+1) k}$. (519)

3. Integrate $\sin. (k x + a)$.

Ans. $-\frac{1}{k} \cos. (k x + a)$. (520)

4. Integrate $\cos. (k x + a)$.

Ans. $\frac{1}{k} \sin. (k x + a)$. (521)

5. Integrate $\text{tang}. k x$. *Ans.* $-\frac{1}{k} \log. \cos. k x$. (522)

6. Integrate $\cot. k x$. *Ans.* $\frac{1}{k} \log. \sin. k x$. (523)

7. Integrate $\sec. k x$. *Ans.* $\frac{1}{2k} \log. \dfrac{1 + \sin. k x}{1 - \sin. k x}$. (524)

8. Integrate $\text{cosec}. k x$. *Ans.* $\frac{1}{k} \log. \text{tang}. \frac{1}{2} k x$. (525)

9. Integrate $e^{a x} \sin. k x$. (526)

Solution. The substitution of (514) in (526), gives by (464),

$$\int . e^{a x} \sin. k x = -\int . e^{a y \sqrt{-1}} \text{Sin}. k y$$

$$= -e^{a y \sqrt{-1}} \frac{a \sqrt{-1}\, \text{Sin}. k y - k\, \text{Cos}. k y}{-(a^2 + k^2)}$$

$$= e^{a x} \frac{a \sin. k x - k \cos. k x}{a^2 + k^2}. \qquad (527)$$

10. Integrate $e^{ax} \cos. k x$.

$$Ans. \quad e^{ax} \frac{a \cos. k x + k \operatorname{Sin}. k x}{a^2 + k^2}. \tag{528}$$

11. Integrate $e^{a \sin. kx} \cos. k x$. $\quad Ans.$ $\frac{1}{ak} e^{a \sin. kx}$. (529)

12. Integrate $e^{a \cos. kx} \sin. k x$. $\quad Ans.$ $-\frac{1}{ak} e^{a \cos. kx}$. (530)

70. *Corollary*. The differential coefficients of (518 – 521, 527, 528), with reference to m, k, and a, give

$$\int. \sin.^m k x . \cos. k x . \log. \sin. k x$$

$$= \left(\log. \sin. k x - \frac{1}{m+1}\right) \frac{\sin.^{m+1} k x}{(m+1) k}, \tag{531}$$

$$\int. \cos.^m k x . \sin. k x . \log. \cos. k x$$

$$= \left(\frac{1}{m+1} - \log. \cos. k x\right) \frac{\cos.^{m+1} k x}{(m+1) k}; \tag{532}$$

$$\int. x \cos. (k x + a) = - d_{c.k} . \tfrac{1}{k} \cos. (k x + a) \tag{533}$$

$$= \frac{1}{k^2} \cos. (k x + a) + \frac{x}{k} \sin. (k x + a),$$

$$(-1)^n \int. x^{2n} \sin. (k x + a) = - d_{c.k}^{2n} . \tfrac{1}{k} \cos. (k x + a), \tag{534}$$

$$(-1)^n \int. x^{2n+1} \cos. (k x + a) = - d_{c.k}^{2n+1} . \tfrac{1}{k} \cos. (k x + a), \tag{535}$$

$$(-1)^n \int. x^{2n} \cos. (k x + a) = \quad d_{c.k}^{2n} . \tfrac{1}{k} \sin. (k x + a), \tag{536}$$

$$(-1)^n \int. x^{2n+1} \sin. (k x + a) = - d_{c.k}^{2n+1} . \tfrac{1}{k} \sin. (k x + a); \tag{537}$$

$$\int. x e^{ax} \sin. k x = d_{c.a} . e^{ax} \frac{a \sin. k x - k \cos. k x}{a^2 + k^2}$$

$$= e^{ax} \left(x - \frac{2a}{a^2 + k^2}\right) \frac{a \sin. k x - k \cos. k x}{a^2 + k^2} + \frac{e^{ax} \sin. k x}{a^2 + k^2}, \tag{538}$$

Trigonometric definite integrals.

$$\int . x^n e^{ax} \sin. kx = d^n_{c.a} . e^{ax} \frac{a \sin. kx - k \cos. kx}{a^2 + k^2}, \qquad (539)$$

$$\int . x^n e^{ax} \cos. kx = d^n_{c.a} . e^{ax} \frac{a \cos. kx + k \sin. kx}{a^2 + k^2}. \qquad (540)$$

71. *Problem.* Find the value of the definite integral

$$\int_{-\infty}^{\infty} . (m x^2 + n) e^{-ax^2 + c} \cos. k x. \qquad (541)$$

Solution. The substitution of $k\sqrt{-1}$ for k in example 4 of § 67, gives for the value of (541)

$$\left(-\frac{m k^2}{4a} + \frac{m}{2a} + n\right) e^{-\frac{k^2}{4a} - c} . \sqrt{\frac{\pi}{a}}. \qquad (542)$$

72. Examples.

1. Find the value of the definite integral

$$\int_{-\infty}^{\infty} . x e^{-(ax^2 + c)} \sin. k x.$$

Ans. $-\frac{k}{2a} e^{-\frac{k^2}{4a} - c} \sqrt{\frac{\pi}{a}}$. (543)

2. Find the values of the definite integrals

$$\int_{-\infty}^{\infty} . x e^{-(ax^2 + c)} \cos. k x,$$

and

$$\int_{-\infty}^{\infty} . (m x^2 + n) e^{-(ax^2 + c)} \sin. k x.$$

Ans. 0. (544)

3. Find the value of the definite integral

$$\int_0^{\infty} . e^{-ax} \sin. k x, \qquad (545)$$

in which a is positive.

Solution. Equation (527) gives, for the value of this integral,

$$\frac{k}{a^2+k^2}. \tag{546}$$

4. Find the value of the definite integral

$$\int_0^\infty . e^{-ax} \cos. k x. \qquad \textit{Ans.} \quad \frac{a}{a^2+k^2}. \tag{547}$$

73. *Problem.* Find the value of the definite integral

$$\int_0^{(2n+\frac{1}{2})\pi} . \sin.^m x . \cos.^p x, \tag{548}$$

in which m, n, and p are positive integers.

Solution. The substitution in (548) of

$$\sin. x = y, \quad x = \sin.^{[-1]} y; \tag{549}$$

whence $\quad \cos. x = (1-y^2)^{\frac{1}{2}}, \; d_{c.y}. x = (1-y^2)^{-\frac{1}{2}}; \quad (550)$

and when $\quad x = 0, \quad y = 0, \quad (551)$

$$x = (2n+\tfrac{1}{2})\pi, \quad y = 1; \tag{552}$$

gives for the value of (548)

$$\int_0^1 . y^m (1-y^2)^{\frac{1}{2}(p-1)}, \tag{553}$$

which may be found by § 44.

74. Examples.

1. Find the value of the definite integrals

$$\int_0^{(2n+\frac{1}{2})\pi} . \sin.^m x$$

and $$\int_0^{(2n+\frac{1}{2})\pi} . \cos.^m x$$

when m is an integer.

Trigonometric definite integrals.

Ans. When m is odd, it is $\frac{2\,.\,4\,.\,.\,.\,.\,.\,(m-1)}{3\,.\,5\,.\,.\,.\,.\,.\,m}$ (554)

when m is even, it is $\frac{1\,.\,3\,.\,.\,.\,.\,.\,(m-1)}{2\,.\,4\,.\,.\,.\,.\,.\,m}\,(2\,n+\tfrac{1}{2})\,\pi.$ (555)

2. Find the value of the definite integral

$$\int_0^{(2\,n+\frac{1}{2})\pi} .\ \sin.\ x\ .\ \cos.\ x. \qquad (556)$$

Ans. 1.

3. Find the value of (548), when m and p are both even.

Ans. $\frac{1.3.5\,.\,.\,.\,.\,.\,(m-1)\times 1.3.5\,.\,.\,(p-1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,.\,(m+p)}\,(2\,n+\tfrac{1}{2})\pi.$ (557)

4. Find the value of (548), when m is even and p odd.

Ans. $\frac{1.3\ 5\,.\,.\,.\,(m-1)\times 2.4.6\,.\,.\,.\,(p-1)}{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,.\,(m+p)},$

or $\frac{2\,.\,4\,.\,6\,.\,.\,.\,(p-1)}{(m+1)\,(m+3)\,.\,.\,.\,.\,(m+p)}.$ (558)

5. Find the value of (548) when m is odd, and p even.

Ans. $\frac{2.4\ 6\,.\,.\,.\,.\,(m-1)\times 1.3.5\,.\,.\,.\,.\,(p-1)}{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,(m+p)},$

or $\frac{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,(m-1)}{(p+1)\,(p+3)\,.\,.\,.\,(m+p)}.$ (559)

6. Find the value of (548), when m and p are both odd.

Ans. $\frac{2.4.6\,.\,.\,.\,(m-1)\times 2.4\ 6\,.\,.\,.\,(p-1)}{2\,.\,4\,.\,6\,.\,.\,.\,.\,.\,.\,.\,(m+p)},$ (560)

or the same with the second answers in (558 and 559).

75. *Problem.* To find the value of the definite integral

$$\int_0^{2n\pi} . \sin.^m x \cos.^p x,$$

in which m, n, and p are positive integers.

Solution. The reduction may be made in this case, precisely as in § 73, it being observed that when

$$x = 0, \quad \text{or} = 2n\pi, \quad y = 0. \tag{561}$$

By this means, the integral, when either m or p is odd, is zero; but, when m and p are both even, it is

$$\frac{1.3.5.....(m-1) \times 1.3\,5....(p-1)}{2.4.6\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,(m+p)} 2n\pi. \tag{562}$$

76. Examples.

1. Find the values of the definite integrals

$$\int_0^{2n\pi} . \sin.^m x$$

$$\int_0^{2n\pi} . \cos.^m x$$

when m is even.

$$Ans. \quad \frac{1.3.5....(m-1)}{2.4.6....\,m} 2n\pi. \tag{563}$$

4. Find the value of the definite integral

$$\int_0^{2n\pi} . \sin. h x . \cos. k x,$$

when h and k are integers. *Ans.* 0. (564)

3. Find the values of the definite integrals

$$\int_0^{2n\pi} . \cos. h\,x \,.\, \cos. k\,x,$$

$$\int_0^{2n\pi} . \sin. h\,x \,.\, \sin. k\,x,$$

when h and k are integers.

Ans. It is zero, unless $\quad h = k,$

in which case, it is $\quad n\,\pi. \qquad$ (565)

CHAPTER VI.

RECTIFICATION OF CURVES.

77. *Problem. To find the length of an arc of a given curve.*

Solution. If s denotes the required arc, its length is readily found by ascertaining the value of its differential coefficient, and integrating it.

Thus if we adopt the notation recently introduced by some of the most eminent mathematicians, and denote the differential coefficient by the capital letter D, and denote by a small letter annexed to D or $\int$, the corresponding independent variable, we have by (570 – 582 of vol. 1),

$$\begin{aligned} s &= \int . Ds = \int_x \sqrt{\,}[1 + (D_x y^2) = \int_x . \sec. \tau = \int_y . \operatorname{cosec}. \tau \\ &= \int_\varphi . \sqrt{\,}[r^2 + (D_\varphi r)^2] = \int_r . \sqrt{\,}[1 + r^2 (D_r \varphi)^2] \\ &= \int_r . \sec. \varepsilon = \int_\varphi . r \operatorname{cosec}. \varepsilon. \end{aligned} \tag{566}$$

76. *Corollary.* The arbitrary constant, which is to be added to complete each of these integrals, corresponds to the indeterminateness of the point at which the measured arc may commence. The condition, by which this point may be determined, will be sufficient to determine the value of the arbitrary constant; or to eliminate it and reduce the result to the form of a definite integral. Thus, if the length of the

arc is required, which extends from the value of x_0 to that of x_1, it is evidently represented by the definite integral

$$s = \int_{x_0}^{x_1} D\, s. \tag{567}$$

78. Examples.

1. Find the length of the arc of the curve of which the equation is

$$y = \tfrac{1}{2}(e^x + e^{-x}).$$

Solution. In this case, we have

$$\begin{aligned}
D\, y &= \tfrac{1}{2}(e^x - e^{-x}) \\
D\, s^2 &= 1 + D\, y^2 = 1 + \tfrac{1}{4}(e^{2x} + e^{-2x}) \\
&= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x}) \\
&= \tfrac{1}{4}(e^x + e^{-x})^2 \\
D\, s &= \tfrac{1}{2}(e^x + e^{-x}) \\
s &= \tfrac{1}{2}(e^x - e^{-x}),
\end{aligned}$$

in which the length of the curve vanishes with $x = 0$.

2. Find the length of the arc of the parabola whose equation is

$$y^2 = 2\, p\, x,$$

counted from the vertex. (568)

Ans. $\tfrac{1}{2}\sqrt{(2\, p\, x + 4\, x^2)} + \tfrac{1}{2}\, p \log. \left[\sqrt{\left(1 + \frac{2\, x}{p}\right)} + \sqrt{\frac{2\, x}{p}}\right].$

3. Find the length of the cycloid from equations (130, 131, of vol. 1), the arc being supposed to commence with x.

Ans. $4\, R\, (1 - \cos. \tfrac{1}{2}\, \theta) = 8\, R \sin.^2 \tfrac{1}{4}\, \theta,$ (569)

and the whole length of a branch is $8\, R$, corresponding to

$$\theta = 2\, \pi. \tag{570}$$

Arc of hyperbolic and logarithmic spiral.

4. Find the length of the hyperbolic spiral, the arc being supposed to commence with $\varphi = \varphi_0$. (571)

$$Ans.\ 2\pi R\left[\sqrt{\left(1+\frac{1}{\varphi_0^2}\right)}-\sqrt{\left(1+\frac{1}{\varphi^2}\right)}+\log.\left(\frac{\varphi+\sqrt{(1+\varphi^2)}}{\varphi_0+\sqrt{(1+\varphi_0^2)}}\right)\right].$$

5. Find the length of the logarithmic spiral, the arc being supposed to commence with r.

$$Ans.\quad r\sec.\,\varepsilon = r\sqrt{\left(1+\frac{1}{\log.^2 a}\right)}. \quad (572)$$

The elliptic and hyperbolic arcs possess some peculiar properties, which deserve particular investigation.

79. *Theorem. The two tangents, which are drawn from a given point to a given ellipse or hyperbola, make equal angles with the two lines which are drawn from the same point to the two foci.*

Thus the two tangents PT and PT' (figs. 1, 2, 3), make equal angles with the lines PF and PF' drawn to the foci; that is, the angles FPT and $F'PT'$ are equal.

Proof. Each of the two tangents PT and PT' is, by examples 2 and 3 of § 131 of vol. 1, equally inclined to the lines drawn from the foci FT and $F'T$, or $F'T$ and FT', so that the angles

$$FTt = F'TP, \text{ and } FT'P = F'T't'.$$

If then the triangles FTP and $F'T'P$ are turned over, around the sides TP and $T'P$, which remain stationary, so as to fall into the positions TPS and $T'PS'$, the points S and S' will be in the lines TF' and $T'F$ produced if necessary. The triangles PSF' and $PS'F$ are, then, equal; for the sides

$$PS = PF,$$
$$PF' = PS';$$

and the side $F'S = FS'$, because each of these two lines is equal to the transverse axis, since in (fig. 1) each is the sum of the two lines FT and $F'T$, or of the two FT' and $F'T'$; while in (figs. 2 and 3) each is the difference of the same two lines. The angles SPF' and FPS' are consequently equal. If the angle FPF' is subtracted from each of these angles (fig. 1), or added to each of them (fig. 2), or diminished by each of them (fig. 3); the resulting angles SPF and $S'PF'$ (figs. 1 and 3), or the excess of 360° over SPF and $S'PF'$ (fig. 2) are equal. Hence FPT and $F'PT'$, which are the halves of SPF and $S'PF'$, are equal.

80. *Corollary.* If an ellipse (fig. 1) or an hyperbola (figs. 2, and 3) be drawn with the points F and F' for foci, and passing through the point P, the tangent to this new curve at the point P will be equally inclined to the two lines PF and PF'; and, therefore, it will also be equally inclined to the two tangents TP and $T'P$.

81. *Theorem. If from any point of the ellipse PP' (fig. 1), or of the hyperbola PP' (fig. 2), which has the points FF' for its foci, tangents are drawn to the ellipse TT' or hyperbola TT' which has the same foci, the sum of the tangents PT and PT' exceeds the included arc TT' by a constant quantity;* that is, by a quantity which is the same, from whatever point of the first ellipse or hyperbola the tangents be drawn.

Proof. Let tangents pt'', pt' be drawn from a second point p infinitely near P. The tangent pt'' exceeds PT by the projection of Pp upon PT diminished by the arc Tt'', or

$$pt'' = PT + PP' \cos. TPS - Tt''.$$

In the same way,

$$p\,t' = P\,T' - P\,P' \cos.\ T'\,P\,S' + T'\,t',$$
$$= P\,T' - P\,P' \cos.\ T\,P\,S + T'\,t';$$

whence

$$p\,t'' + p\,t' = P\,T + P\,T' + (T'\,t' - T\,t'').$$

But $$t''\,t' = T\,T' + (T'\,t' - T\,t'');$$

and, therefore,

$$(p\,t'' + p\,t') - t''\,t' = (P\,T + P\,T') - T\,T'.$$

The excess of the sum of the tangents over the included arc does not, then, increase by moving the point P a small distance upon the curve, in which it is situated; and consequently this excess must be a constant quantity.

82. *Corollary.* Had an hyperbola $P\,Q$ (fig. 1), or an ellipse $P\,Q$ (fig. 2), been drawn, with the foci F and F', it might easily have been shown in the same way, that the excess of the difference of the tangents $P\,T$ and $P\,T'$ over the difference of the arcs $Q\,T$ and $Q\,T'$ was constant. But as the point P, in moving along the curve $P\,Q$, approaches Q, the tangents and arcs decrease, and finally vanish when P coincides with Q. At the point Q, therefore, the excess of the difference of the tangents over the difference of the arcs is nothing, and therefore this excess is nothing for every point of the curve $P\,Q$.

Hence, *if from any point P of the hyperbola $P\,Q$ (fig. 1), or of the ellipse $P\,Q$ (fig. 2), which has the points F and F' for its foci, tangents are drawn to the ellipse $T\,T'$ (fig. 1), or to the hyperbola $T\,T'$ (fig. 2), which has the same foci, the difference of the tangents $P\,T$ and $P\,T'$ is equal to the difference of the arcs $Q\,T$ and $Q\,T'$.*

83. *Corollary.* If the excess of the sum of the tangents $P\,T$ and $P\,T'$ over the arc $T\,T'$ is denoted by $2\,E$, the two preceding theorems give

$$P\,T + P\,T' = Q\,T + Q\,T' + 2\,E$$
$$P\,T' - P\,T = Q\,T' - Q\,T;$$

whence

$$P\,T' = Q\,T' + E$$
$$P\,T = Q\,T + E. \tag{573}$$

84. *Corollary.* Upon the transverse axis $A\,A'$ (fig. 4) of the ellipse, describe the semicircumference $A\,L\,L'\,A'$, draw the ordinates $L\,T\,M$ and $L'\,T'\,M'$, and join $O\,L$, $O\,L'$, O being the common centre of the ellipse and circle. Let, if $O\,B$ is the semiconjugate axis,

$$\left.\begin{array}{ll} \varphi = L\,O\,B, & \varphi' = L'\,O\,B, \\ A = O\,A\ , & B = O\,B, \\ x = O\,M\ , & x' = O\,M', \\ y = M\,T\ , & y' = M'\,T', \\ z = M\,L\ , & z' = M'\,L', \\ s = B\,T\ , & s' = B\,T', \\ \multicolumn{2}{c}{e = 1 - \dfrac{B^2}{A^2},} \end{array}\right\} \tag{574}$$

we have, by section 163 of vol. 1, and by the triangles $L\,O\,B$, $L'\,O\,B$,

$$x = A \sin.\,\varphi, \qquad x' = A' \sin.\,\varphi', \tag{575}$$

$$y = \frac{B}{A}\,.\,z = \frac{B}{A}\,.\,A \cos.\,\varphi = B \cos.\,\varphi, \quad y' = B \cos.\,\varphi'; \tag{576}$$

and, by differentiation, letting φ be the independent variable,

$$D\,x = A\cos.\,\varphi, \qquad D\,y = B\sin.\,\varphi. \tag{577}$$

$$\begin{aligned} D\,s^2 &= A^2\cos.^2\varphi + B^2\sin.^2\varphi \\ &= A^2 + (B^2 - A^2)\sin.^2\varphi \\ &= A^2(1 - e^2\sin.^2\varphi), \end{aligned} \tag{578}$$

$$D\,s = A\sqrt{(1 - e^2\sin.^2\varphi)}; \tag{579}$$

$$s = \int A\sqrt{(1 - e^2\sin.^2\varphi)}. \tag{580}$$

This integral is one of a class which are called *elliptic integrals*. There are three integrals in this class, and the present one is said to be of the *second order*. The following notation has been universally adopted.

$$\Delta\,(e\,\varphi) = \sqrt{(1 - e^2\sin.^2\varphi,)} \tag{581}$$

or the Δ may be used without the $e\,\varphi$, when there is no danger of confusion,

$$E\,.\,\varphi = \int_0^{\varphi}\Delta, \tag{582}$$

or

$$E\,.\,(e\,\varphi) = \int_0^{\varphi}\Delta\,(e\,\varphi). \tag{583}$$

Hence

$$s = A\,E\,.\,\varphi. \tag{584}$$

85. *Corollary*. Let the two tangents $L\,R$ and $L'\,R$, drawn to the circle at the points L and L', meet at R. By reducing all the ordinates of the circle in the ratio of B to A, the circle is changed into the ellipse. By the same system of reduction, the lines $R\,L$ and $R\,L'$ will be changed into other straight lines $P\,T$ and $P\,T'$, and these will be tangent to the ellipse, because the points T and T' are the only ones whose ordinates will be as small as the corresponding ordinates of the ellipse. By joining $O\,R$, the right triangles $O\,R\,L$, $O\,R\,L'$, and $O\,R\,N$ give

Elliptic integral of the second order.

$$OR = \frac{OL}{\cos.\, LOR} = \frac{A}{\cos.\, \frac{1}{2}(\varphi - \varphi')}, \tag{585}$$

$$ON = OR \cos.\, RON = \frac{A \sin.\, \frac{1}{2}(\varphi + \varphi')}{\cos.\, \frac{1}{2}(\varphi - \varphi')}, \tag{586}$$

$$RN = OR \sin.\, RON = \frac{A \cos.\, \frac{1}{2}(\varphi + \varphi')}{\cos.\, \frac{1}{2}(\varphi - \varphi')},$$

$$PN = \frac{B}{A} \cdot RN = \frac{B \cos.\, \frac{1}{2}(\varphi + \varphi')}{\cos.\, \frac{1}{2}(\varphi - \varphi')}. \tag{587}$$

The condition that the point P is upon an ellipse or an hyperbola, as in fig. 1, of which the semiaxes are A' and B, gives, by putting

$$a = \left(\frac{A}{A'}\right)^2, \quad \pm b = \left(\frac{B}{B'}\right)^2, \tag{588}$$

where the upper sign corresponds to the ellipse and the lower to the hyperbola, the equation

$$\frac{a \sin.^2 \frac{1}{2}(\varphi + \varphi')}{\cos.^2 \frac{1}{2}(\varphi - \varphi')} + \frac{b \cos.^2 \frac{1}{2}(\varphi + \varphi')}{\cos.^2 \frac{1}{2}(\varphi - \varphi')} = 1, \tag{589}$$

or

$$a \sin.^2 \tfrac{1}{2}(\varphi + \varphi') + b \cos.^2 \tfrac{1}{2}(\varphi + \varphi') = \cos.^2 \tfrac{1}{2}(\varphi - \varphi'), \tag{590}$$

and

$$a - a \cos.\,(\varphi + \varphi') + b + b \cos.\,(\varphi + \varphi') = 1 + \cos.\,(\varphi - \varphi'), \tag{591}$$

$$\begin{aligned} a + b - 1 &= (a - b) \cos.\,(\varphi + \varphi') + \cos.\,(\varphi - \varphi') \\ &= (a - b + 1) \cos.\,\varphi \cos.\,\varphi' + (b - a + 1) \sin.\,\varphi \sin.\,\varphi'. \end{aligned} \tag{592}$$

If the point P were taken so that T' coincided with B, φ' would be zero, and if the corresponding value of φ is denoted by φ_0, (592) gives

$$a + b - 1 = (a - b + 1) \cos.\,\varphi_0; \tag{593}$$

Elliptic integral of the second order.

which, substituted in (592), divided by $a-b+1$, gives

$$\cos.\ \varphi_0 = \cos.\ \varphi \cos.\ \varphi' + \frac{b-a+1}{a-b+1} \sin.\ \varphi \sin.\ \varphi'. \quad (594)$$

But the condition that the given ellipse has the same foci with the curve in which P is situated, gives

$$A^2 - B^2 = A'^2 \mp B'^2 = \frac{A^2}{a} - \frac{B^2}{b}, \quad (595)$$

$$A^2\left(\frac{1}{a}-1\right) = B^2\left(\frac{1}{b}-1\right),$$

$$e^2 = 1 - \frac{B^2}{A^2} = 1 - \frac{b\,(1-a)}{a\,(1-b)} = \frac{a-b}{a\,(1-b)}. \quad (596)$$

But, by (593),

$$\sin.^2\ \varphi_0 = 1 - \left(\frac{a+b-1}{a-b+1}\right)^2 = \frac{4\,a\,(1-b)}{(a-b+1)^2} \quad (597)$$

$$e^2 \sin.^2\ \varphi_0 = \frac{4\,(a-b)}{(a-b+1)^2},$$

$$\Delta^2_{\varphi_0} = \frac{(b-a+1)^2}{(a-b+1)^2}, \qquad \Delta\,\varphi_0 = \pm\frac{b-a+1}{a-b+1}; \quad (598)$$

which, substituted in (594), gives

$$\cos.\ \varphi_0 = \cos.\ \varphi \cos.\ \varphi' \pm \sin.\ \varphi \sin.\ \varphi'\ \Delta\,.\,\varphi_0. \quad (599)$$

And this equation of condition is the same, with the condition that the point P of fig. 1 is upon an ellipse or hyperbola having the same foci with the given ellipse, the upper sign of (599) *corresponding to the ellipse, and the lower to the hyperbola.*

86. *Corollary.* If a spherical triangle (fig. 5) be drawn, of which the sides are φ, φ' and φ_0, and the opposite angles θ, θ', θ_0, we have by (331) of Spherical Trigonometry,

Elliptic integral of the second order.

$$\cos.\,\theta_0 = \frac{\cos.\,\varphi_0 - \cos.\,\varphi\,\cos.\,\varphi'}{\sin.\,\varphi\,\sin.\,\varphi'}\,; \tag{600}$$

whence by (599),

$$\cos.\,\theta_0 = \pm\,\varDelta\,\varphi_0\,. \tag{601}$$

Hence

$$\cos.^2\,\theta_0 = 1 - e^2\,\sin.^2\,\varphi_0\,, \tag{602}$$

$$e\,\sin.\,\varphi_0 = \surd\,(1 - \cos.^2\,\theta_0) = \sin.\,\theta_0\,; \tag{603}$$

and therefore

$$e = \frac{\sin.\,\theta_0}{\sin.\,\varphi_0} = \frac{\sin.\,\theta}{\sin.\,\varphi} = \frac{\sin.\,\theta'}{\sin.\,\varphi'}\,. \tag{604}$$

This equation gives

$$\varDelta\,\varphi = \surd(1 - e^2\,\sin.^2\,\varphi) = \surd(1 - \sin.^2\,\theta) = \mp\cos.\,\theta, \tag{605}$$

$$\varDelta\,\varphi' = \surd(1 - e^2\,\sin.^2\,\varphi') = \surd(1 - \sin.^2\,\theta') = \quad \cos.\,\theta'. \tag{606}$$

The upper signs in (601) and (605) correspond to the case in which the curve is an ellipse, and the lower to the case in which it is an hyperbola. The signs in (605 and 606) are derived from the consideration that when φ' is zero, φ is equal to φ_0; but when φ is zero, we have, by using the signs as before, (607)

$$\varphi' \pm \varphi_0 = 0. \tag{608}$$

87. *Corollary*. If we make (fig. 4)

$$\left.\begin{array}{l} t = P\,T, \quad t' = P\,T', \\ \psi = \text{the inclination of } P\,T \text{ to the axis } A\,A' \\ \psi' = \text{the inclination of } P\,T' \text{ to } A\,A'. \end{array}\right\} \tag{609}$$

The inclination of $L\,R$ to $A\,A'$ is φ, and that of $L'\,R$ is φ'; hence

$$\left.\begin{array}{l} M\,N = L\,R\,\cos.\,\varphi = A\,\cos.\,\varphi\,\tan.\,\tfrac{1}{2}\,(\varphi - \varphi') \\ M'\,N = L'\,R\,\cos.\,\varphi' = A\,\cos.\,\varphi'\,\tan.\,\tfrac{1}{2}\,(\varphi - \varphi') \end{array}\right\} \tag{610}$$

Elliptic integral of the second order.

$$t = \frac{MN}{\cos.\psi} = A \tan.\tfrac{1}{2}(\varphi - \varphi')\,.\,\frac{\cos.\varphi}{\cos.\psi} \tag{611}$$

$$t = \frac{M'N}{\cos.\psi'} = A \tan.\tfrac{1}{2}(\varphi - \varphi')\,.\,\frac{\cos.\varphi'}{\cos.\psi'}\,. \tag{612}$$

But the construction of TP gives

$$\tan.\psi = \frac{B}{A}\tan.\varphi, \quad \tan.\psi' = \frac{B}{A}\tan.\varphi'\,; \tag{613}$$

whence

$$\frac{1}{\cos.\psi} = \sqrt{(1+\tan.^2\psi)} = \sqrt{\left(1+\frac{B^2}{A^2}\tan.^2\varphi\right)}$$
$$= \sqrt{(1+\tan.^2\varphi - e^2\tan.^2\varphi)} = \sqrt{(\sec.^2\varphi - e^2\tan.^2\varphi)}$$
$$= \sec.\varphi\sqrt{(1-e^2\sin.^2\varphi)} = \sec.\varphi\,\Delta\,.\,\varphi, \tag{614}$$
$$= \mp \sec.\varphi\cos.\theta.$$

In the same way,

$$\frac{1}{\cos.\psi'} = \frac{\Delta\,\varphi'}{\cos.\varphi'} = \frac{\cos.\theta'}{\cos.\varphi'}\,, \tag{615}$$

which, substituted in (611 and 612), give

$$t = A\tan.\tfrac{1}{2}(\varphi-\varphi')\,\Delta\,\varphi = \mp A\tan.\tfrac{1}{2}(\varphi-\varphi')\cos.\theta \tag{616}$$

$$t' = A\tan.\tfrac{1}{2}(\varphi-\varphi')\,\Delta\,\varphi' = \quad A\tan.\tfrac{1}{2}(\varphi-\varphi')\cos.\theta'. \tag{617}$$

88. *Corollary.* When φ' is zero, (616 and 617) become

$$t_0 = \pm \tan.\tfrac{1}{2}\varphi_0\cos.\theta_0 \tag{618}$$

$$t'_0 = A\tan.\tfrac{1}{2}\varphi_0. \tag{619}$$

89. *Corollary.* If the semicircle ARA' with its tangents RL or RL' were turned round AA' as an axis, so as to be brought above the plane of the ellipse, until the angle which the two planes made with each other were one, whose cosine

was equal to the quotient of B divided by A, the ellipse and its tangents would evidently be the projections of the circle and its tangents. The angles which t and t' made with the tangents to the circle of which they were the projections, would evidently from (616 and 617) be θ, and θ'. A consideration of the spherical right triangles formed at the points of meeting S and S' of the tangents, would lead anew to the same equations which we have already obtained.

90. *Corollary*. The equations (616 and 617) give, by using the signs as in (607),

$$t' \pm t = A \tan. \tfrac{1}{2} (\varphi - \varphi') (\cos. \theta' - \cos. \theta)$$
$$= 2 A \tan. \tfrac{1}{2} (\varphi - \varphi') \sin. \tfrac{1}{2} (\theta' + \theta) \sin. \tfrac{1}{2} (\theta - \theta'). \quad (620)$$

But by (350) of Trigonometry, and fig. 5,

$$\sin. \tfrac{1}{2} (\theta' + \theta) : \sin. \tfrac{1}{2} (\theta - \theta') = \tan. \tfrac{1}{2} \varphi_0 : \tan. \tfrac{1}{2} (\varphi - \varphi'), \quad (621)$$

$$\text{or } \sin. \tfrac{1}{2} (\theta' + \theta) \tan. \tfrac{1}{2} (\varphi - \varphi') = \sin. \tfrac{1}{2} (\theta - \theta') \tan. \tfrac{1}{2} \varphi_0 ; \quad (622)$$

which, substituted in (620), gives

$$t' \pm t = 2 A \tan. \tfrac{1}{2} \varphi_0 \sin.^2 \tfrac{1}{2} (\theta - \theta')$$
$$= A \tan. \tfrac{1}{2} \varphi_0 [1 - \cos. (\theta - \theta')]. \quad (623)$$

In the same way,

$$t_0' \pm t_0 = A \tan. \tfrac{1}{2} \varphi_0 (1 + \cos. \theta_0). \quad (624)$$

Hence

$$t_0' - t' \pm (t_0 - t) = A \tan. \tfrac{1}{2} \varphi_0 [\cos. (\theta - \theta') + \cos. \theta_0]. \quad (625)$$

But by (319) of Trigonometry,

$$\cos. \theta_0 = -\cos. (\theta - \theta') + 2 \sin. \theta \sin. \theta' \cos.^2 \tfrac{1}{2} \varphi_0 , \quad (626)$$

Sum of elliptic integrals of the second order.

which, substituted in (625), gives by means of (604),

$$t_0'-t'\pm(t_0-t)=2\,A\tan.\tfrac{1}{2}\varphi_0\cos.^2\tfrac{1}{2}\varphi_0\sin.\theta\sin.\theta'$$
$$=A\sin.\varphi_0\sin.\theta\sin.\theta'$$
$$=A\,e^2\sin.\varphi_0\sin.\varphi\sin.\varphi'. \qquad (627)$$

91. *Corollary.* If T_0 (fig. 1) is the point for which φ becomes φ_0, we have

$$\left.\begin{aligned} A\,E\,\varphi &= T\,B = s, \\ A\,E\,\varphi' &= T'\,B = s' \\ A\,E\,\varphi_0 &= T_0\,B = s_0 \\ A\,E\,\varphi_1 &= Q\,B = s_1. \end{aligned}\right\} \qquad (628)$$

When the point P at which the tangents meet are situated upon the secondary ellipse, we have

$$T\,T' = s - s'; \qquad (629)$$

and because the excess of the sum of the tangents over the included arc is constant,

$$t+t'-(s-s')=t_0+t_0'-s_0;$$

or
$$s_0-s+s'=t_0+t_0'-(t+t'). \qquad (630)$$

Hence, by (627 and 628),

$$E\,\varphi_0+E\,\varphi'-E\,\varphi=e^2\sin.\varphi_0\sin.\varphi'\sin.\varphi, \qquad (631)$$

in which φ_0, φ' and φ are subject to the condition (599), identical with one of the following conditions, easily deduced from the spherical triangle of fig. 5, by means of (605) and (606);

$$\cos.\varphi=\cos.\varphi_0\cos.\varphi'-\sin.\varphi_0\sin.\varphi'\,\Delta\,\varphi, \qquad (632)$$
$$\cos.\varphi'=\cos.\varphi_0\cos.\varphi+\sin.\varphi_0\sin.\varphi\,\Delta\,\varphi'. \qquad (633)$$

But if the point P is situated upon the hyperbola, we have

$$Q\,T=s-s_1,\quad Q\,T'=s_1-s',\quad Q\,T_0=s_0-s_1; \qquad (634)$$

and because the excess of the difference of the tangents over the difference of the arcs counted from Q is constant,

$$t' - t + s - 2\,s_1 + s' = t'_0 - t_0 + s_0 - 2\,s_1,$$

$$s + s' - s_0 = t'_0 - t_0 - t' + t. \qquad (635)$$

Hence, by (627 and 628),

$$E\,\varphi + E\,\varphi' - E\,\varphi_0 = e^2 \sin.\varphi \sin.\varphi' \sin.\varphi_0, \qquad (636)$$

in which φ, φ' and φ_0 are subject to the condition (599), which is identical with (633), or with the following condition;

$$\cos.\varphi = \cos.\varphi_0 \cos.\varphi' + \sin.\varphi' \Delta\,\varphi. \qquad (637)$$

92. *Corollary.* The proposition contained in (636) is, evidently, the same with that of (631). It follows, therefore, that the point of meeting of the tangents drawn at the extremities of the arc $s_0 - s$ of (635) is upon an ellipse, which passes through the point of meeting of the tangents drawn to the extremities of the arc s', and which has the same foci with the given ellipse; the same may be inferred with regard to the tangents drawn through the extremities of the arcs $s_0 - s'$ and s of (635). It follows, in the same way, that the point of meeting of the tangents drawn at the extremities of the arc $s_0 - s'$ of (630), is upon the hyperbola which passes through the point of meeting of the tangents drawn at the extremities of the arc s, and which has the same foci with the given ellipse.

93. *Corollary.* When the points T and T' coincide at the point Q, (636) gives

$$2\,E\,\varphi_1 = e^2 \sin.^2 \varphi_1 \sin.\varphi_0 + E\,\varphi_0. \qquad (638)$$

94. *Corollary.* The supplements of the angles $\pi - \theta$, $\pi - \theta'$, $\pi - \theta_0$ of the triangle of fig. 5, may be the sides of a spherical

Elliptic integral of first order.

triangle, of which, $\pi - \varphi$, $\pi - \varphi'$, $\pi - \varphi_0$ are the opposite angles. In this case, since

$$\sin.(\pi - \theta) = \sin.\theta \text{ and } \cos.(\pi - \theta) = -\cos.\theta, \text{ \&c.} \tag{639}$$

and

$$\frac{1}{e} = \frac{\sin.\varphi}{\sin.\theta} = \frac{\sin.\varphi'}{\sin.\theta'} = \frac{\sin.\varphi_0}{\sin.\theta_0}; \tag{640}$$

we have, by putting

$$\varDelta'.\theta = \varDelta.\left(\frac{1}{e}.\theta\right) = \sqrt{\left(1 - \frac{1}{e^2}\sin.^2\theta\right)}, \tag{641}$$

$$E'.\theta = \int_0^\theta \varDelta'.\theta; \tag{642}$$

$$E'.\theta + E'.\theta' - E'.\theta_0 = \frac{1}{e^2}\sin.\theta\sin.\theta'\sin.\theta''$$

$$= e\sin.\varphi\sin.\varphi'\sin.\varphi''. \tag{643}$$

But since the differentiation of

$$e\sin.\varphi = \sin.\theta$$

gives

$$D_\varphi\theta. = \frac{e\cos.\varphi}{\cos.\theta} = \frac{e\cos.\varphi}{\varDelta.\varphi}, \tag{645}$$

we have

$$E'.\theta = \int_0^\theta \varDelta'.\theta = \int_0^\theta.\cos.\varphi$$

$$= \int_0^\varphi \frac{e\cos.^2\varphi}{\varDelta\varphi}. \tag{646}$$

We have also, by (581),

$$\cos.^2\varphi = 1 - \sin.^2\varphi = 1 - \frac{1}{e^2} + \frac{1}{e^2}(\varDelta\varphi)^2, \tag{647}$$

which, substituted in (646), gives

$$E'.\theta = -\frac{1-e^2}{e}\int_0^\varphi \frac{1}{\varDelta.\varphi} + \frac{1}{e}\int_0^\pi \varDelta\varphi$$

$$= -\frac{1-e^2}{e}\int_0^\varphi \frac{1}{\varDelta\varphi} + \frac{1}{e}.E.\varphi. \tag{648}$$

Elliptic integral of first order.

Similar equations may be found for $E' \theta'$ and $E' \theta_0$, all of which, substituted in (643), give

$$-\frac{1-e^2}{e}\left[\int_0^{\varphi}\frac{1}{\Delta\varphi}+\int_0^{\varphi'}\frac{1}{\Delta\varphi}+\int_0^{\varphi_0}\frac{1}{\Delta\varphi}\right]$$

$$+\frac{1}{e}(E.\varphi+E.\varphi'-E.\varphi_0)=e\sin.\varphi\sin.\varphi'\sin.\varphi''; \quad (649)$$

whence, by (636),

$$\int_0^{\varphi}\frac{1}{\Delta\varphi}+\int_0^{\varphi'}\frac{1}{\Delta\varphi}-\int_0^{\varphi_0}\frac{1}{\Delta\varphi}=0. \quad (650)$$

95. *Corollary.* The integral

$$\int_0^{\varphi}\frac{1}{\Delta.\varphi} \quad (651)$$

is the *elliptic integral of the first kind*, and is denoted by $F.\varphi$, that is,

$$F.\varphi=\int_0^{\varphi}\frac{1}{\Delta.\varphi} \quad \text{or} \quad F(e.\varphi)=\int_0^{\pi}\frac{1}{\Delta(e.\varphi)}. \quad (652)$$

Hence, by (650),

$$F.\varphi+F.\varphi'-F.\varphi_0=0, \quad (653)$$

where φ, φ' and φ_0 are subjected to the same conditions as in (636).

96. *Corollary.* In the same way in which φ is connected with φ' by means of the construction of fig. 1, in which P is upon the ellipse, giving by (653 and 631) the equation

$$F.\varphi=F.\varphi'+F.\varphi_0, \quad (654)$$

other points, φ'', φ''', &c. might be found, such that

$$F\varphi'=F.\varphi''+F.\varphi_0$$

$$F.\varphi''=F.\varphi'''+F.\varphi_0, \ \&c; \quad (655)$$

Elliptic integral of first order.

whence
$$F.\varphi = F\varphi'' + 2\,F.\varphi_0$$
$$= F.\varphi''' + 3\,F.\varphi_0$$
$$= F.\varphi_n + n\,F.\varphi_0\,; \tag{656}$$
or
$$F.\varphi - F\varphi_n = n\,(F.\varphi - F\varphi'). \tag{657}$$

97. *Corollary.* If in equation (656) φ_n vanishes, (656) becomes
$$F.\varphi = n\,.\,F\varphi_0\,, \tag{618}$$
and it is obviously easy to obtain a geometrical construction of the corresponding condition between φ and φ_0, by taking several successive tangents, BTP, $PT'P'$, &c. as in (fig. 5), and the points P, P', P'', &c., correspond to φ, φ', φ'', &c. The tangents might also be drawn to the circle, from the successive points R, R', R'', &c., of the ellipse, of which the semiaxes are A' and $\frac{A\,B'}{B}$. A similar construction, in which the series of tangents does not commence with B, would satisfy the conditions of (656 and 657.)

98. *Corollary.* The value of φ for the arc $B\,T$ (fig. 4) is equal to the angle which the corresponding tangent $L\,S$ to the circle makes with the transverse axis $A\,A'$. Denote by χ the angle which the tangent $S\,T$ to the ellipse makes with the conjugate axis, so that
$$\tau = \tfrac{1}{2}\,\pi - \psi. \tag{659}$$
When the plane of the circle is elevated above that of the ellipse, the spherical right triangle formed about S for the centre of the sphere, has φ for its hypothenuse, and θ and τ for its legs. It is represented by (fig. 6), and if α is the angle opposite to θ, and ν the angle opposite to τ, we have

Complete integral. Complementary functions.

$$\cos.\,\alpha = \frac{B}{A} = \sqrt{(1-e^2)} = \cos.\,\theta \sin.\,\nu = \cot.\,\varphi \cot.\,\psi \quad (660)$$

$$\sin.\,\alpha = e, \quad (661)$$

$$\cos.\,\psi = \sin.\,\varphi \sin.\,\nu, \quad (662)$$

$$\cos.\,\nu = \sin.\,\alpha \sin.\,\psi = e \sin.\,\psi. \quad (663)$$

These equations give by the differentiation of (660), by representing by F_1 the value of F corresponding to a right angle, and observing that when φ is zero, τ is also zero, and ψ a right angle; but that when φ is a right angle, τ is also a right angle, and ψ is zero,

$$\Delta\,(e\,.\,\varphi) = \cos.\,\theta = \frac{\cos.\,\alpha}{\sin.\,\nu}, \quad (664)$$

$$D\psi\,.\,\varphi = -\frac{\cos.\,\alpha \sin.^2 \varphi}{\cos.^2 \psi}, \quad (665)$$

$$\frac{D\psi\,.\,\varphi}{\Delta(e\,.\,\varphi)} = -\frac{\sin.\,\nu \sin.^2 \varphi}{\cos.^2 \psi} = -\frac{\sin.\,\varphi}{\cos.\,\psi} = -\frac{1}{\sin.\,\nu}$$

$$= -\frac{1}{\sqrt{(1-\cos.^2 \nu)}} = -\frac{1}{\sqrt{(1-e^2 \sin.^2 \psi)}} = -\frac{1}{\Delta(e\,.\,\psi)}; \quad (666)$$

$$F.\,\varphi = \int_0^{\varphi} \frac{1}{\Delta\,(e\,.\,\varphi)} = -\int_{\psi}^{\frac{1}{2}\pi} \frac{D\psi\,.\,\varphi}{\Delta\,(e\,.\,\varphi)}$$

$$= \int_{\psi}^{\frac{1}{2}\pi} \frac{1}{\Delta(e\,.\,\psi)} = \int_0^{\frac{1}{2}\pi} \frac{1}{\Delta(e\,.\,\psi)} - \int_0^{\psi} \frac{1}{\Delta(e\,.\,\psi)}$$

$$= F_1 - F\,(e\,.\,\psi); \quad (667)$$

so that $F\varphi$ and $F\psi$ are two functions whose sum is the function F_1 which is called the *complete integral*, and the two functions are called *complementary* with regard to each other, as well as the angles φ and ψ, upon which they depend.

99. *Corollary.* The three angles ψ, ψ' and ψ_0, which correspond to φ, φ' and φ_0, satisfy the equation, equivalent to (653),

$$F.\psi + F.\psi' - F.\psi_0 = 0, \tag{668}$$

when they are subject to the condition, equivalent to (599),

$$\cos.\psi_0 = \cos.\psi \cos.\psi' - \sin.\psi \sin.\psi' \,\Delta\psi_0. \tag{669}$$

100. *Corollary.* Denote by 2χ the angle, which $F\,T$ (fig. 1) makes with the transverse axis. The angle, which $F\,T$ makes with the tangent, is

$$\tfrac{1}{2}\pi + \psi - 2\chi = \tfrac{1}{2}\pi - (2\chi - \psi). \tag{670}$$

The projection of $F\,T$ upon the tangent is, therefore,

$$F\,T \times \sin.(2\chi - \psi), \tag{671}$$

while that of $F'\,T$ upon the same tangent is

$$F'\,T \times \sin.(2\chi - \psi); \tag{672}$$

the sum of which is

$$(F\,T + F'\,T)\sin.(2\chi - \psi) = 2a\sin.(2\chi - \psi). \tag{673}$$

But this is the projection of $F\,F' = 2ae$ upon the same tangent, which is

$$2ae\sin.\psi; \tag{674}$$

and therefore, we have the equation

$$\sin.(2\chi - \psi) = e\sin.\psi; \tag{675}$$

whence, by (663),

$$2\chi - \psi = \tfrac{1}{2}\pi - \nu. \tag{676}$$

The equation (675) gives by putting

$$e'' = \frac{2\sqrt{e}}{1+e}, \tag{677}$$

$$\Delta''\varphi = \Delta(e'', \varphi), \tag{678}$$

$$F''\varphi = F(e'', \varphi); \tag{679}$$

Transformation of elliptic integrals.

$$\tan.\psi = \frac{\sin. 2\chi}{e+\cos. 2\chi}, \frac{1}{\cos.\psi} = \frac{\sqrt{((e+1)^2-2e\sin.^2\chi)}}{e+\cos. 2\chi} = \frac{(e+1)\varDelta''.\chi}{e+\cos. 2\chi}, \tag{680}$$

$$\varDelta\psi = \cos.(2\chi-\psi) = \frac{1+e\cos. 2\chi}{e+\cos. 2\chi}.\cos.\psi, \tag{681}$$

$$D_\chi.\psi = \frac{2\cos.(2\chi-\psi)}{\cos.(2\chi-\psi)+e\cos.\psi}, \tag{682}$$

$$\frac{D_\chi.\psi}{\varDelta\psi} = \frac{2}{\cos.(2\chi-\psi)+e\cos.\psi} = \frac{2(e+1)\varDelta''.\chi}{1+e^2+2e\cos. 2\chi}$$

$$= \frac{2(e+1)\varDelta''.\chi}{(e+1)^2(\varDelta''.\chi)^2} = \frac{2}{(e+1)\varDelta''.\chi}; \tag{683}$$

$$F.\psi = \int_0^\psi \frac{1}{\varDelta\psi} = \int_0^\chi \frac{D_\chi.\psi}{\varDelta\psi} =$$

$$= \int_0^\chi \frac{2}{(e+1)\varDelta''.\chi} = \frac{2}{e+1}F''.\chi = \frac{e''}{\sqrt{e}}F''.\chi. \tag{684}$$

101. *Corollary.* If χ, χ', χ_0 correspond to ψ, ψ', ψ_0, we have the equation

$$F''\chi + F''\chi' - F''\chi_0 = 0, \tag{685}$$

with the condition that

$$\cos.\chi_0 = \cos.\chi\cos.\chi' - \sin.\chi\sin.\chi'\varDelta''\chi_0. \tag{686}$$

102. *Corollary.* In the same way in which $F'\chi$ is obtained from $F\psi$, another function $F''\chi_1$ might be obtained from $F'\chi$, and so on, until a series was obtained in which the values of e, e'', e'''', &c. form a series of eccentricities, in which each differs less and less from unity. It may be shown that e'' differs from unity less than e does, for (677) gives

$$1-e'' = \frac{(1-\sqrt{e})^2}{1-e} = \frac{(1-e)^2}{(1+\sqrt{e})^2(1+e)} = \frac{1-e}{(1+\sqrt{e})^2(1+e)}(1-e), \tag{689}$$

in which the factor of $1 - e$ is evidently less than unity, and decreases rapidly with the decrease of $1 - e$. The value of $F.\varphi$ may therefore be made to depend upon a value of $F = (e_n . \varphi_n)$, in which e_n differs from unity by as small a quantity as we please, and we have

$$F.\varphi = \frac{e'' e''' \ldots \ldots e_n}{\sqrt{(e\,e'' \ldots \ldots)}} F(e_n \varphi_n) = \sqrt{\frac{e_n}{e^2}} \sqrt{(e\,e'' \ldots \ldots e_n)}\, F(e_n\, \varphi_n). \tag{690}$$

103. *Corollary*. In the same way $F(e\varphi)$, by reversing the above process, may be made to depend upon the value of $F(e_n\, \varphi_n)$, in which e_n is as small as we please. In this case e'' may be found from e by reversing the accents in (677) and solving the equation with regard to e'', which gives

$$e'' = \left(\frac{1 - \sqrt{(1 - e^2)}}{e}\right)^2 = \left(\frac{e}{1 + \sqrt{(1 - e^2)}}\right)^2. \tag{691}$$

104. *Corollary*. If we put in (677)

$$e = \tan.^2 \tfrac{1}{2} \beta, \tag{692}$$

we have

$$e'' = \sin.\, \beta. \tag{693}$$

105. *Corollary*. We have by (681, 680, 683),

$$\cos.\, 2\,\chi = \frac{\cos.\, \psi - e\, \varDelta\, \psi}{\varDelta\, \psi - e \cos.\, \psi} \tag{694}$$

$$e + \cos.\, 2\,\chi = \frac{(1 - e^2) \cos.\, \psi}{\varDelta\, \psi - e \cos.\, \psi} = (\varDelta\, \psi + e \cos.\, \psi) \cos.\, \psi \tag{695}$$

$$\varDelta''\, \chi = \frac{\varDelta\, \psi + e \cos.\, \psi}{1 + e},$$

$$E''.\, \chi = \int_0^{\chi} \varDelta''\, \chi = \int_0^{\psi} \varDelta''\, \chi\, .\, D_\psi\, \chi = \int_0^{\psi} . \frac{\frac{1}{2}(e+1)(\varDelta''.\chi)^2}{\varDelta\, \psi}$$

Reduction of elliptic integrals.

$$=\int_0^{\psi}\frac{(\Delta\psi+e\cos.\psi)^2}{2(1+e)\Delta\psi}=\int_0^{\psi}\frac{2(\Delta\psi)^2+2e\cos.\psi\,\Delta\psi-(1-e^2)}{2(1+e)\Delta\psi}$$

$$=\int_0^{\psi}\frac{\Delta\psi}{1+e}-\int_0^{\psi}\frac{1-e}{2\Delta\psi}+\int_0^{\psi}\frac{e\cos.\psi}{1+e}$$

$$=\frac{E\psi}{1+e}-\tfrac{1}{2}(1-e)F\psi+\frac{e\sin.\psi}{1+e}, \tag{696}$$

which may serve to deduce the value of $E(e\,.\,\varphi)$ from those of $E\,.\,(e_n\,\varphi_n)$, in which e_n is very small, or differs but little from unity.

106. *Corollary*. Potential functions may be applied to the hyperbola very nearly in the same way in which circular functions have been applied to the ellipse. Thus if we put

$$A\,\mathrm{Cos.}\,\varphi=x,\quad B\,\mathrm{Sin.}\,\varphi=y, \tag{697}$$

x and y are the coördinates of the hyperbola, of which the equation is

$$\left(\frac{x}{A}\right)^2-\left(\frac{y}{B}\right)^2=1. \tag{698}$$

The length of the hyperbolic arc is, by putting

$$e=\sqrt{\left(1+\frac{A^2}{B^2}\right)}, \tag{699}$$

$$s=B\int\sqrt{(1+e^2\,\mathrm{Sin.}^2\,\varphi)}.$$

If we let

$$\digamma(e\,\varphi)=\sqrt{(1+e^2\,\mathrm{Sin.}^2\,\varphi)} \tag{700}$$

$$\mathfrak{E}(e\varphi)=\int_0^{\varphi}\digamma(e\,\varphi), \tag{701}$$

we have

$$s=B\,\mathfrak{E}(e\varphi). \tag{702}$$

107. *Corollary*. The condition that the point of contact in (fig. 2) is upon another hyperbola which has the same foci with

the given hyperbola, is expressed algebraically by the equation

$$\text{Cos.}\,\varphi_0 = \text{Cos.}\,\varphi\ \text{Cos.}\,\varphi' - \text{Sin.}\,\varphi\ \text{Sin.}\,\varphi' . \sqrt{}\,\varphi_0\,, \qquad (703)$$

and corresponds to the equation

$$\mathfrak{E}\,\varphi' + \mathfrak{E}\,\varphi - \mathfrak{E}\,\varphi_0 = e^2\,\text{Sin.}\,\varphi_0\ \text{Sin.}\,\varphi\ \text{Sin.}\,\varphi', \qquad (704)$$

in which φ', φ_0, φ, correspond respectively to the last and first points of the hyperbolic arch included by the two tangents, and to the last point of the arc of which the first point is the vertex.

108. *Corollary.* In the same way, if we put

$$\mathfrak{F}\,\varphi = \int_0^\varphi \frac{1}{\sqrt{}\,\omega}, \qquad (705)$$

we have, with the condition (703), the equation

$$\mathfrak{F}\,\varphi' + \mathfrak{F}\,\varphi - \mathfrak{F}\,\varphi_0 = 0. \qquad (706)$$

109. *Corollary.* When e was changed into its reciprocal in § 94, it became greater than unity, and ceased, therefore, to correspond to an ellipse. It may easily be shown that, in this case, however, the transverse axis and the angle φ become imaginary of the form $A\sqrt{-1}$ and $\varphi\sqrt{-1}$; so that the ellipse changes into the hyperbola, of which B is the transverse axis, and A the conjugate axis; and the circular change into potential functions. This case is, therefore, the same as the one just investigated, and it may be remarked, that the equilateral hyperbola takes the place of the circle.

110. *Corollary.* If θ is determined by the condition that

$$\cos.\,\theta = \text{Tan.}\,\varphi, \qquad (707)$$

Length of hyperbolic arch.

we have $\sin.\,\theta = \frac{1}{\text{Cos.}\,\varphi}$, $\cot.\,\theta = \text{Sin.}\,\varphi$; (708)

whence $D_\theta . \varphi = -\operatorname{cosec.}\theta$

$$\digamma . \varphi = \sqrt{(1+e^2\cot.^2\theta)} = \frac{e\sqrt{(1-\frac{e^2-1}{e^2}\sin.^2\theta)}}{\sin.\,\theta} \tag{709}$$

$$= e\operatorname{cosec.}\theta\, \Delta\,.\,(e'''.\,\theta), \tag{710}$$

if $$e''' = \sqrt{\frac{e^2-1}{e^2}}. \tag{711}$$

Hence

$$\mathcal{I}\,.\,\varphi = \int_0^\varphi . \frac{1}{\digamma\,\varphi} = -\int_\theta^{\frac{1}{2}\pi} \frac{D_\theta\,\varphi}{\digamma\,\varphi}$$

$$= e.\int_\theta^{\frac{1}{2}\pi} \frac{1}{\digamma\,.\,(e'''.\,\theta)} = e[F(e'''.\,\tfrac{1}{2}\pi) - F(e'''.\,\theta)]. \tag{712}$$

111. *Corollary.* Let ω be so taken that

$$\text{Sin.}\,\varphi = \frac{1}{e}\tan.\,\omega; \tag{713}$$

and we have, by putting

$$e' = \sqrt{(1-\frac{1}{e^2})}, \tag{714}$$

$$\text{Cos.}\,\varphi = \frac{\sqrt{(1-e'^2\sin.^2\omega)}}{\cos.\,\omega} = \frac{\Delta\,(e'\,\omega)}{\cos.\,\omega}$$

$$= \frac{\Delta'\,\omega}{\cos.\,\omega}, \tag{715}$$

$$\digamma\,\varphi = \frac{1}{\cos.\,\omega}, \tag{716}$$

$$D_\omega\,.\,\varphi = \frac{1}{e\,\Delta'\,\omega\,.\,\cos.\,\omega}. \tag{717}$$

Hence the length of the arc of the hyperbola is

$$s = B\,\text{Ɐ}.\varphi = B\int_0^\varphi D_\omega\,\varphi\,.\,\digamma\,.\,\varphi$$

$$= \frac{B}{e}\int_0^\omega \frac{1}{\cos.^2\omega\,\Delta'\omega}$$

$$= \frac{B}{e}\int_0^\omega \frac{1}{\Delta'\omega} + \frac{B}{e}\int_0^\omega \frac{\sin.^2\omega}{\cos.^2\omega\,\Delta'\omega}$$

$$= \frac{B}{e}F'\omega + \frac{B}{e\,(1-e'^2)}\int_0^\omega \frac{(1-e'^2)\tan.^2\omega}{\Delta'\omega} \tag{718}$$

$$= \frac{B}{e}F'\omega - B\,e\,E'\omega + B\,e\int_0^\omega\left(\Delta'\omega + \frac{(1-e'^2)\tan.^2\omega}{\Delta'\omega}\right),$$

But, we have,

$$\Delta'\omega + \frac{(1-e'^2)\tan.^2\omega}{\Delta'\omega} = \frac{\Delta'\omega}{\cos.^2\omega} + \left[\frac{1-e'^2}{\Delta'\omega} - \Delta'\omega\right]\tan.^2\omega$$

$$= \frac{\Delta'\omega}{\cos.^2\omega} - \frac{e'^2\sin.^2\omega}{\Delta'\omega}$$

$$= D\,.(\tan.\,\omega\,\Delta'\omega); \tag{719}$$

which, substituted in (718), gives

$$s = \frac{B}{e}F'.\omega - B\,e\,E'\omega + B\,e\tan.\,\omega\,\Delta'\omega$$

$$= \sqrt{(A^2+B^2)}\left[\frac{F'.\omega}{e^2} - E'\omega + \tan.\,\omega\,\Delta'\omega\right]. \tag{720}$$

112. Examples.*

1. Prove that the tangent let fall from the centre of the hyperbola upon the tangent is in the notation of § 111.

$$B\, e \tan. \omega\ \varDelta' \omega = \sqrt{(A^2 + B^2)} \tan. \omega\ \varDelta' \omega \tag{721}$$

2. Prove that if e'' and χ are so taken that

$$e'' = \frac{2\sqrt{e'}}{1+e'}, \tag{722}$$

$$\sin. (2\chi - \omega) = e' \sin. \omega, \tag{723}$$

the length of the arc of the hyperbola is

$$s = \sqrt{(A^2+B^2)} [E'\omega - 2(1+e') E''\chi + 2 e' \sin. \chi + \tan. \omega\ \varDelta' \omega]. \tag{724}$$

113. *Definition. A curve of double curvature* is a curve all the parts of which do not lie in the same plane.

114. *Problem. To find the length of an arc of a curve of double curvature.*

Solution. Since the length of an infinitesimal arc of the curve is equal to the distance apart of the two infinitely near points x, y, z and $x + dx$, $y + dy$, $z + dz$; or, algebraically,

$$d s = \sqrt{(dx^2 + dy^2 + dz^2)}; \tag{725}$$

the length of an arc is

$$s = \int \sqrt{(D x^2 + D y^2 + D z^2)}. \tag{726}$$

* The examples will not hereafter be strictly confined to the subject of the chapter, but will extend to any exercises suggested by the investigations in the chapter.

115. Examples.

1. To find the length of an arc of the helix.

Solution. The helix is a curve formed by a string wrapped round a cylinder in such a way as to make a constant angle with the side of the cylinder. Hence, if the plane of $x\,y$ is that of the base of the cylinder, if the centre of the base is the origin of coördinates, and if

$$\left.\begin{array}{l} \varrho = \text{radius of the base of the cylinder} \\ \varphi = \text{the angle which the projection of radius vector on the plane of } x\,y \text{ makes with the axis of } x, \\ \alpha = \text{the angle which the string makes with the side of the cylinder,} \end{array}\right\} \qquad (727)$$

the equations of the helix are

$$x = \varrho \cos.\varphi, \quad y = \varrho \sin.\varphi, \quad z = \varrho\,\varphi \cot.\alpha. \qquad (728)$$

Hence

$$s = \int_0^\varphi \varrho \sqrt{(1 + \cot.^2\alpha)}$$

$$= \frac{\varrho\,\varphi}{\sin.\alpha}. \qquad (729)$$

2. To find the length of the arc of the curve formed by winding a string round a solid of revolution, in such a way as to make a constant angle with the meridional arc of the surface. The *meridional arc of a surface of revolution* is the intersection of the surface with a plane passing through the axis of revolution.

Ans. With the notation of the preceding example, the length of the arc is

$$s = \sec \alpha \int_z . \sqrt{[1+(D_z\, \varrho)^2]} = s' \sec. \alpha; \qquad (730)$$

where $s' =$ the arc of the meridional section, corresponding to the required arc s.

The angle φ may be substituted for z in (730) by means of the equation

$$\varphi = \tan. \alpha \int_z \frac{D_z\, s'}{\varrho} \,. \qquad (731)$$

3. To find the length of the arc in Example 2, when the solid is a right cone.

Ans. If $\beta =$ the angle which the side of the cone makes with the axis,

$z_0 =$ the value of z at the beginning of the arc

$$s = (z - z_0) \sec. \alpha \sec. \beta$$

$$= \sec. \alpha \sec. \beta \,.\, e^{\varphi \sin. \beta \cot. \alpha} \qquad (732)$$

4. To find the length of the arc in Example 2, when the solid is a sphere; that is, to find the length of an arc of a *rhumb line.*

Ans. If $R =$ the radius of the sphere

$\theta =$ the inclination of the radius vector to the axis of z,

$\theta_0 =$ the value of θ at the beginning of the arc, (733)

the length of the arc is

$$s = R\, (\theta - \theta_0) \sec. \alpha \qquad (734)$$

and φ may be substituted for θ by means of the equation

$$\varphi = \log. \tan. \tfrac{1}{2}\, \theta - \log. \tan. \tfrac{1}{2}\, \theta_0 \,. \qquad (735)$$

5. To find the length of the arc in Example 2, when the solid is an ellipsoid of revolution.

$$\left.\begin{array}{l} \textit{Ans.} \text{ If } A = \text{the semi-transverse axis,} \\ e = \text{the excentricity,} \\ \theta \quad \text{corresponds to } \varphi \text{ in (574),} \end{array}\right\} \quad (736)$$

the length of the arc is

$$s = A \sec. \alpha\,(E\,\theta - E\,\theta_0), \qquad (737)$$

and φ may be substituted for θ by means of the equation

$$\varphi = \log. \frac{(\varDelta\,\theta - \cos.\,\theta)\sin.\,\theta_0}{(\varDelta\,\theta_0 - \cos.\,\theta_0)\sin.\,\theta} + e \log. \frac{e\cos.\,\theta + \varDelta\,\theta}{e\cos.\,\theta_0 + \varDelta\,\theta_0}, \qquad (738)$$

in the case of the oblate ellipsoid. But in case of the prolate ellipsoid, the equation is

$$\varphi = \log. \frac{\varDelta\,\theta + \sqrt{(1-e^2)}\sin.\,\theta}{\varDelta\,\theta_0 + \sqrt{(1-e^2)}\sin.\,\theta_0} + \log. \frac{\cos.\,\theta_0}{\cos.\,\theta}$$

$$+ \frac{e}{\sqrt{(1-e^2)}} \operatorname{tang.}^{[-1]} \frac{e(\sin.\,\theta - \sin.\,\theta_0)}{1 - \sin.\,\theta - \sin.\,\theta_0 + 2\sin.\,\theta\sin.\,\theta_0}. \qquad (739)$$

116. *Problem. To find the shortest line, which can be drawn, subject to given conditions.*

Solution. The variation in the length of the curve arising from any very small change in its equation, which may be made, consistently with its conditions, is in general proportional to the magnitude in the change of the equation; it must, therefore, change its sign with the change of sign in the variation of the equation. Thus if δx denote the variation of the equation, we shall have for the variation in the length of the curve,

Shortest line.

$$\delta s = D_x s \,.\, \delta x + \tfrac{1}{2} D_x^2 s \,.\, \delta x^2 + \&c. \qquad (740)$$

the second member of which is reduced to its first term when δx is an infinitesimal. But, in the case of a maximum or minimum, δs should not change its sign with δx, and therefore this first term should be wanting, that is, the variation of s should be zero, while the second variation of s should not vanish; or if the second variation should chance to vanish with the first, the third variation must also vanish. The principles of finding this class of maxima or minima are the same with those of B. II. Chap. VIII. But the process of finding a maximum or minimum of a definite integral, such as

$$\int_{s_0}^{s_1} D s, \qquad (741)$$

dependent upon a variable function, is quite different from that for the ordinary maximum or minimum; and such problems are often considered by themselves, under the title of the *Method of Variations.*

The function (741) can vary in two ways, either by the variation of the functions upon which s depends, or by the variation of the limits of the integration. The condition that it is a maximum, is, therefore, expressed by the equation

$$\delta \int_{s_0}^{s_1} D s = \delta s_1 - \delta s_0 + \int_{s_0}^{s_1} \delta D s = 0. \qquad (742)$$

The value of $D s$ usually depends upon many variables, t, x, y, &c., and their differential coefficients, in such a way that, if t is taken for the independent variable, any change in the functions by which x, y, &c. depend upon t, gives the equations

$$\delta s_1 = D s_1 \,.\, \delta t_1, \quad \delta s_0 = D s_0 \,.\, \delta t_0, \qquad (743)$$

$$\delta D s = X \delta x + Y \delta y + \&c. \; + X' \delta D x + Y' \delta D y + \&c.$$
$$+ X'' \delta D^2 x + Y'' \delta D^2 y + \&c.; \qquad (744)$$

which, substituted in (742), give

$$D s_1 \delta t_1 - D s_0 \delta t_0 + \int_{t_0}^{t_1} (X \delta x + X' \delta D x$$
$$+ X'' \delta D^2 x + \&c. \;\; + Y \delta y + \&c.) = 0. \qquad (745)$$

But, by (262),

$$\int . X' \delta D x = \int . X' D \delta x = X' \delta x - \int D X' . \delta x \qquad (746)$$
$$\int . X'' \delta D^2 x = \int . X'' D^2 \delta x = X'' D \delta x - \int D X'' . D \delta x$$
$$= X'' D \delta x - D X'' . \delta x + \int D^2 X'' . \delta x, \&c. \qquad (747)$$

The terms in the last members of (746 and 747), which are not under the sign of integration, must, in passing to the definite integrals, be referred to the limits of integration But it must be observed, that the variations in (746 and 747) are taken upon the supposition that the independent variable t does not itself vary, and that only the functions vary, by which x, y, &c. are connected with it; whereas the limits of integration may themselves necessarily vary with a change of this function, and therefore t_0 and t_1 are supposed to vary. If, then, $\delta' x_0$, $\delta' y_0$, &c. denote the variations arising from the change of the functions, the values of the complete variations are

$$\delta x = \delta' x_0 + D x_0 . \delta t_0, \; \&c. \qquad (748)$$

whence $$\delta' x_0 = \delta x_0 - D x_0 . \delta t_0, \; \&c. \qquad (749)$$

Hence (746 and 747) give

$$\int_{t_0}^{t_1} X' \delta D x = X'_1 \delta' x_1 - X'_0 \delta' x_0 - \int_{t_0}^{t_1} D X' . \delta x \qquad (750)$$

$$\int_{t_0}^{t_1} X'' \delta D^2 x = X''_1 D \delta' x_1 - D X''_1 \delta' x_1 - X''_0 D \delta' x_0$$
$$- D X'_0 \delta' x_0 + \int_{t_0}^{t_1} D^2 X'' . \delta x, \&c. \qquad (751)$$

in which δ' is given by (749).

Method of Variations.

These equations, substituted in (745), give

$$Ds_1\,\delta t_1 - Ds_0\,\delta t_0 + (X_1 - DX_1' + \&c.)\,\delta' x_1 - (X_0 - DX_0' + \&c.)\,\delta' x_0$$
$$+ (X_1' - \&c.)\,D\,\delta'.\,x_1 - (X_0' - \&c)\,D\,\delta' x_0 + \&c.$$
$$+ \int_{t_0}^{t_1}.\,[(X - DX' + D^2X'' - \&c.)\,\delta x + (Y - \&c.)\,\delta y + \&c.] = 0. \qquad (752)$$

The terms of (752), which are under the sign of integration, express a variation which belongs to each point of the curve independently of all the other points, and which must, therefore, be equal to zero for each point; which gives the general equation

$$(X - D\,X' + D^2\,X'' - \&c)\,\delta x + \&c. = 0. \qquad (753)$$

The variables, t, x, y, &c. may be bound together by some conditions, represented by the equations

$$L = 0, \quad M = 0, \qquad (754)$$

in which L, M, may be functions of t, x, y, &c. The variations of these equations will then give linear equations between δx, δy, &c. from which the values of some of the variations δx, δy, &c. can be determined in terms of the others. These values, substituted in (753), will reduce the number of variations in (753) to the smallest possible number, and those which remain will be wholly independent of each other, and therefore their coefficients must vanish. The equations, thus obtained from making these coefficients equal to zero, will be the required equations of the shortest time.

If, in addition to the equations (754), the limits of the curve are subject to peculiar conditions; these conditions, with those of (754), referred to the limits of the curve, may be combined with the terms of (752), which are not under the sign of integration, and the equations for determining the extreme points

of the curve may be found by the same method by which the equations of the curve itself are found.

117. *Corollary.* The preceding process for finding the minimum of (741), may be applied to finding the maximum or minimum of any definite integral, such as

$$\int_{t_0}^{t_1} . V, \tag{756}$$

by changing in the various formulæ $D s$ into V.

118. *Corollary.* The number of the variations δx, δy, &c. determined by (754), is plainly equal to the number of the equations of (754). The number of the variations left undetermined, therefore, in (753), and consequently the number of equations obtained from (753), is equal to the number of the variations not determined by (754). The whole number of equations then of the required curve, is equal to the whole number of the variables x, y, z, &c., among which the independent variable is not included; that is, there are just as many equations as are required to determine the curve.

In the same way, it may be shown that there are just enough equations to determine the extreme points of the curve.

119. *Corollary.* The following method of eliminating the variations from (753), which are determined by (754), is more symmetrical than the usual one, which is proposed in § 116. Multiply the variation of each of the equations (754) by some quantity, such as λ, μ, &c. and add the sum of all the products to (753). The values of λ, μ, &c. may be determined by putting equal to zero, the coefficients of just as

many of the variations δx, δy, &c. The substitution of these values of λ, μ, &c. in the other coefficients, will reduce (753) to an equation, from which as many of the variations have disappeared as there are equations (754). The remaining coefficients, being those of independent variations, must therefore be equal to zero; that is, *each of the coefficients in the sum formed by the addition of the products of* (754) *by* λ, μ, &c. *to* (753), *may be put equal to zero, and* λ, μ, *&c. may be eliminated from the result by the usual process.*

120. *Corollary.* If all the variables had been, in the outset, eliminated from $D\,s$ (741) or V (756), which could have been eliminated by means of the equations (754), the remaining ones would have been independent of each other, and would have given, at once, from (753),

$$\left.\begin{array}{l} X - D\,X' + D^2\,X'' - \&c. = 0, \\ Y - D\,Y' + D^2\,Y'' - \&c. = 0, \quad \&c. \end{array}\right\} \quad (757)$$

If, moreover, certain of the variables, and among them the independent variable, had been taken so as to be the very functions of the variables which were constant under the additional conditions at one of the limits, as that of t_0; we should have for those variables

$$\delta\,t_0 = 0, \ \&c. \qquad (758)$$

and there would have been no additional conditions between $\delta\,x_0$, $\delta\,y_0$, &c., which in this case would not differ from $\delta'\,x_0$, $\delta'\,y_0$, &c.; so that (752) would give

$$X'_0 - D\,X''_0 + \&c. = 0, \quad Y'_0 - D\,Y''_0 + \&c. = 0, \ \&c. \qquad (759)$$

$$X''_0 - \&c. = 0 \qquad Y''_0 - \&c. = 0, \ \&c. \qquad (760)$$

of which (759) are the same with (757) referred to this ex-

tremity. The equations (758), however, involve the hypothesis that there is no condition, by which both extremities are bound together.

121. *Corollary.* If the curve is referred to rectangular coördinates x, y, z, of which x is the independent variable, we have

$$D s = \sqrt{(1 + D y^2 + D z^2)}, \tag{761}$$

$$\delta D s = \frac{D y}{D s} \delta D y + \frac{D z}{D s} \delta D z ; \tag{762}$$

whence

$$\left.\begin{array}{l} Y' = \dfrac{D y}{D s}, \quad Z' = \dfrac{D z}{D s}, \\ Y = 0, \quad Y'' = 0, \text{ \&c.} \end{array}\right\} \tag{763}$$

These equations, substituted in (753 and 752), give

$$D \left(\frac{D y}{D s}\right) . \delta y + D \left(\frac{D z}{D s}\right) . \delta z = 0, \tag{764}$$

$$D s_1 \, \delta x_1 - D s_0 \, \delta x_0 + \frac{D y_1}{D s_1} \delta' y_1 + \frac{D z_1}{D s_1} \delta' z_1$$

$$- \frac{D y_0}{D s_0} \delta' y_0 - \frac{D z_0}{D s_0} \delta' z_0 = 0. \tag{765}$$

122. *Corollary.* Any condition between the rectangular coördinates of the preceding article must be expressed by an algebraical equation, which may be regarded as the *equation of a surface upon which the shortest possible line is to be drawn.*

If the positions of the axes of coördinates are so taken that one of the axes, that of z for instance, is perpendicular to the

Line of least curvature.

surface at one of the points x, y, z, through which the curve passes, we should have, at this point,

$$\delta z = 0, \tag{766}$$

and, therefore, by (764),

$$D\left(\frac{Dy}{Ds}\right) = 0. \tag{767}$$

But the plane of the axes of x and y is, at this point, parallel to the tangent plane, and

$$\frac{Dy}{Ds}$$

is the cosine of the inclination of the curve to the axis of y; so that by (767) this inclination is constant.

Hence the direction of the shortest line drawn upon a surface, has at each point no curvature in the direction of the tangent plane; it has, then, less curvature at each point than any other curve drawn upon the same surface through that point, and having the same tangent with it; that is, *it has the maximum radius of curvature of all lines which have a common tangent, and are drawn upon the same surface;* it coincides, then, with the direction of a riband which is wound round the surface in such a way as to bend only towards the surface, without bending in the tangent plane either to the right or left.

123. *Corollary.* Upon any surface whatever, such as a cylinder or cone, formed by the bending of a plane, and which is designated as *a developable surface, the shortest line becomes a straight line when the surface is bent back into a plane;* and it may be remarked, that

any surface formed by the motion of a straight line, which remains in two successive positions in the same plane, is developable.

124. *Corollary.* It is obvious, from (353 of vol. 1), that the hyperboloid of equation (350 of vol. 1) is not developable, and that therefore the preceding corollary is not applicable to it, although it may be generated by the motion of a straight line.

125. *Corollary.* The shortest curve upon the sphere is an arc of a great circle.

126. *Corollary.* The curve drawn upon the surface of the earth, upon the principles of § 122, is called *the geodetical curve, and, therefore, this is the shortest curve which can be drawn upon the earth's surface.*

127. *Corollary.* If the position of the axes is taken as in § 122, but with the condition that the axis of x shall be the normal to the surface, we have for the point x_0, y_0, z_0,

$$\delta x_0 = 0; \tag{768}$$

whence, by (765),

$$D y_0 \,\delta y_0 + D z_0 \,\delta z_0 = 0, \tag{769}$$

$$\frac{\delta y_0}{\delta z_0} = -\frac{D z_0}{D y_0}. \tag{770}$$

Since the point x_0, y_0, z_0, is upon the given surface, any additional condition for this point would be equivalent to requiring it to be upon some other surface, so that it would have to be at the common intersection of the two surfaces. The first member of (770) expresses, then, the tangent of the angle which this intersection makes with the axis of z, while the second member expresses the negative of the cotangent of the angle which the required curve makes with the same axis.

The shortest curve which can be drawn upon a given surface from one curve upon that surface to another curve, or to a point upon the surface, *is perpendicular at either extremity to the limiting curve at that extremity.*

128. *Corollary.* If the required line is subject to no condition except at its extremities, the variations in (764) are entirely independent, which gives the equations

$$D\left(\frac{Dy}{Ds}\right)=0, \qquad D\left(\frac{Dz}{Ds}\right)=0; \tag{771}$$

that is, since the direction of the axes is wholly arbitrary, the required line has no curvature in any direction, and *is, consequently, a straight line.*

If the extremity x_0, y_0, z_0, is subject to a condition, that it must be upon a given surface; the normal to that surface at the extremity of the line may be taken for the axis of x_0, which gives

$$\delta x_0 = 0, \tag{772}$$

and leaves δy_0 and δz_0 arbitrary; whence, by (765),

$$\frac{Dy_0}{Ds_0}=0, \qquad \frac{Dz_0}{Ds_0}=0; \tag{773}$$

that is, the cosine of the angle which the required line makes with the surface in each direction is zero; or, in other words, the required line is perpendicular to the surface.

If the extremity x_0, y_0, z_0, is subject to two conditions, that is, if it is at the intersection of two given surfaces; let this line be the axis of z_0, and we have

$$\delta x_0 = 0, \quad \delta y_0 = 0, \tag{774}$$

whence (765) gives

$$\frac{D z_0}{D s_0} = 0, \tag{775}$$

or the required line is perpendicular to the given line. Hence

The shortest line which can be drawn between two given surfaces, or two given lines, or a line and a surface, or a point and a surface, or a point and a line, is the *straight line which is perpendicular to the surface or line at the corresponding extremity.*

129. *Corollary.* If the shortest line is required to be drawn upon a surface of revolution, let the axis of z be the axis of revolution, let u be the projection of the radius vector upon the plane of $x\,y$, and let φ be the angle which u makes with the axis of x; and we have, by taking z for the independent variable,

$$D s = \sqrt{(u^2 D \varphi^2 + D u^2 + 1)}. \tag{776}$$

But by the equation of the surface, u is a given function of z, and, therefore, not subject to variation. Hence

$$\delta D s = \frac{u^2 D \varphi D \delta \varphi}{D s}. \tag{777}$$

The equation gives, then,

$$D \frac{u^2 D \varphi}{D s} = 0, \tag{778}$$

the integral of which is

$$\frac{u^2 D \varphi}{D s} = C; \tag{779}$$

in which C is an arbitrary constant, and the independent variable may be any variable whatever, because it is only the ratio of two differential coefficients which enters into (779).

Geodetic curve upon the oblate ellipsoid.

130. Examples.

1. To find the shortest line which can be drawn upon the oblate ellipsoid of revolution.

Solution. Let A be the greater, and B the smaller semi-axis of the generating ellipse, and e the eccentricity; we have for the equations of the ellipse, as in (575 and 576),

$$x = A \sin. \theta, \quad y = B \cos. \theta\,; \tag{780}$$

and x in this equation is the same with u in (779), and y is the same with z. Hence (776 and 779) give, by taking θ for the independent variable, and using the notation of elliptic integrals,

$$D\, s^2 = A^2 \sin.^2 \theta\, D\, \varphi^2 + A^2 \cos.^2 \theta + B^2 \sin.^2 \theta$$

$$= A^2 (\sin.^2 \theta\, D\, \varphi^2 + \Delta\, \theta^2) = \frac{A^4 \sin.^4 \theta}{C^2}\, D\, \varphi^2, \tag{781}$$

$$D\, \varphi = \frac{C\, \Delta\, \theta}{\sin. \theta \sqrt{(A^2 \sin.^2 \theta - C^2)}}\,. \tag{782}$$

Let α, ψ, and e be taken, so that

$$C = A \sin. \alpha, \tag{783}$$

$$\cos. \psi = \cos. \theta \sec. \alpha, \tag{784}$$

$$e' = \frac{e \cos. \alpha}{\sqrt{(1 - e^2 \sin.^2 \alpha)}}\,; \tag{785}$$

and we have, by taking ψ for the independent variable,

$$\sin. \theta\, D_\psi\, \theta = \cos. \alpha \sin. \psi, \tag{786}$$

$$\Delta\, \theta^2 = 1 - e^2 (1 - \cos.^2 \alpha \cos.^2 \psi) = 1 - e^2 (\sin.^2 \alpha + \cos.^2 \alpha \sin.^2 \psi)$$

$$= (1 - e^2 \sin.^2 \alpha)\, (1 - e'^2 \sin.^2 \psi)$$

$$= (1 - e^2 \sin.^2 \alpha)\, \Delta'\, \psi^2, \tag{787}$$

$$\sqrt{(A^2 \sin.^2 \theta - C^2)} = A \sqrt{(\sin.^2 \theta - \sin.^2 \alpha)}$$

$$= A \sqrt{(\cos.^2 \alpha - \cos.^2 \theta)} = A \cos. \alpha \sin. \psi, \tag{788}$$

$$D_\psi \varphi = \frac{C \Delta \theta \, D_\psi \theta}{\sin. \theta \sqrt{(A^2 \sin.^2 \theta - C^2)}} = \frac{\sin. \alpha \sqrt{(1 - e^2 \sin.^2 \alpha)} . \Delta' \psi}{\sin.^2 \theta}$$

$$= \frac{\sqrt{(1 - e^2 \sin.^2 \alpha)} \, \Delta' \psi}{\sin. \alpha (1 + \cot.^2 \alpha \sin.^2 \psi)} = \frac{\sqrt{(1 - e^2 \sin.^2 \alpha)} \, \Delta' \psi^2}{\sin. \alpha (1 + \cot.^2 \alpha \sin.^2 \psi) \, \Delta' \psi} \tag{789}$$

$$= \frac{\sin. \alpha}{\cos.^2 \alpha \sqrt{(1 - e^2 \sin.^2 \alpha)}} \left[\frac{1 + e^2 \cos. 2\alpha}{(1 + \cot.^2 \alpha \sin.^2 \psi) \, \Delta' \psi} - \frac{e^2 \cos.^2 \alpha}{\Delta' \psi} \right].$$

Hence, if we adopt the notation

$$\Pi (n . e \psi) = \int_0^\psi \frac{1}{(1 + n \sin.^2 \psi) \, \Delta \psi}, \tag{790}$$

and put

$$n = \cot.^2 \alpha, \tag{791}$$

(789) gives

$$\varphi = \frac{\sin. \alpha}{\cos.^2 \alpha \sqrt{(1 - e^2 \sin.^2 \alpha)}} \tag{792}$$

$$[(1 + e^2 \cos. 2 \alpha) (\Pi(n \, e' \psi_1) - \Pi (n \, e' \psi_0)) - e^2 \cos.^2 \alpha \, (F' \psi_1 - F' \psi_0)],$$

which is the required equation of the curve.

The length of the curve becomes, by substitution in (781),

$$s = A \sqrt{(1 - e^2 \sin.^2 \alpha)} \, (E' \psi_1 - E' \psi_0). \tag{793}$$

The integral (790) is called the *elliptic integral of the third order*, and admits of theorems similar to those of the first and second orders.

2. To find the shortest line upon the prolate ellipsoid.

Ans. Let the axes of the ellipsoid be represented by the same letters, as in the preceding example; and let the equations of the ellipsoid be

$$\left.\begin{array}{ll} u = B\cos.\theta, & z = A\sin.\theta, \\ C = B\cos.\alpha, & \sin.\theta = \sin.\alpha\sin.\psi, \\ e' = e\sin.\alpha, & n = -\sin.^2\alpha, \end{array}\right\} \quad (794)$$

the equation of the required curve is (795)

$$\varphi = \cos.\alpha\sqrt{(1-e^2)}[\varPi(n\,e'\,\psi_1) - \varPi(n\,e'\,\psi_0)] + \frac{e^2\cos.\alpha}{\sqrt{(1-e^2)}}(F\psi_1 - F\psi_0),$$

and the length of the curve is

$$s = A\,(E'\psi_1 - E'\psi_0). \quad (796)$$

3. Prove that if φ_0, φ and φ' satisfy the lower equation (599), and if

$$N = \sqrt{[n\,(n+1)\,(n+e^2)]} \quad (797)$$

the elliptic integrals of the third order will satisfy the equation

$$\varPi\,(n\,e\,\varphi') + \varPi\,(n\,e\,\varphi) - \varPi\,(n\,e\,\varphi_0)$$
$$= \frac{n}{N}\tan.^{[-1]}\left(\frac{N\sin.\varphi'\sin.\varphi\sin.\varphi_0}{1 + n\,(1 - \cos.\varphi'\cos.\varphi\cos.\varphi_0)}\right). \quad (798)$$

CHAPTER VII.

QUADRATURE OF SURFACES.

131. *Problem. To find the quadrature of a surface.*

Solution. Let $M\,L\,M'\,L'$ (fig. 7) be the portion of the surface, whose area is to be found, and which may be either plane or curved. Let the conditions of the bounding line be expressed by an equation between two variables, l and m. Suppose two lines, $L\,L'$, $l\,l'$, drawn infinitely near each other, and in such a way that l is constant throughout the extent of these lines; and let the lines $M\,M'$, $m\,m'$, be so drawn that m is constant throughout their extent. If then σ is taken to denote the required surface, we have

$$D_l\,\sigma = \frac{\text{the area } L\,L'\,l\,l'}{d\,l} = \frac{L}{d\,l} \tag{799}$$

and

$$D^2_{l\,.\,m}\,\sigma = \frac{D_m\,L}{d\,l} = \frac{\text{the area } a\,b\,c\,d}{d\,l\,.\,d\,m}. \tag{800}$$

But, if

$$\left.\begin{aligned} a &= \text{the angle } b\,a\,c, \\ s' &= \text{an arc of } L\,L', \\ s'' &= \text{an arc of } M\,m; \end{aligned}\right\} \tag{801}$$

we have

$$\text{the arc } a\,b = d\,s', \text{ the arc } a\,c = d\,s'', \tag{802}$$
$$\text{the area } a\,b\,c\,d = \sin.\,a\,d\,s'\,.\,d\,s''$$

and, since m is the only variable in s', and l the only variable in s'',

$$D^2_{l\,.\,m}\,\sigma = \sin.\,a\,D_m\,s'\,.\,D_l\,s'', \tag{803}$$

and the accents may be omitted in (802) without any ambiguity. Hence

$$\sigma = \int_l \int_m \sin. a\, D_m . s . D_l . s\, ; \qquad (804)$$

in which $D_m s$ and $D_l s$ may be taken directly from the general expression for $D s$, and a is the inclination of two lines drawn through a point, in such a way, that for the one l is constant, and for the other m is constant.

132. *Corollary.* When the surface is plane, (570) of vol. 1 gives for rectangular coördinates,

$$D_x s = 1, \qquad D_y s = 1, \qquad (805)$$

and it is obvious that a is a right angle; whence

$$\sigma = \int_x \int_y . 1 = \int_y . x = \int_x . y, \qquad (806)$$

or supplying the place of arbitrary constants by the form of definite integrals,

$$\sigma = \int_{x_0}^{x_1} \int_{y_0}^{y_1} 1 = \int_{x_0}^{x_1} (y_1 - y_0) = \int_{y_0}^{y_1} (x_1 - x_0), \qquad (807)$$

in which the values of $x_0\ x_1\ y_0\ y_1$, are determined by the bounding curve.

133. *Corollary.* When the surface is plane, (574) of vol. 1 gives

$$D_\varphi s = r, \qquad D_r s = 1, \qquad (808)$$

and a is a right angle; whence

$$\sigma = \int_\varphi \int_r . r = \int_r . r\,\varphi = \tfrac{1}{2} \int_\varphi . r^2; \qquad (809)$$

or

$$\sigma = \int_{\varphi_0}^{\varphi_1} \int_{r_0}^{r_1} . r = \int_{r_0}^{r_1} . (r_1\,\varphi_1 - r_0\,\varphi_0) = \tfrac{1}{2} \int_{\varphi_0}^{\varphi_1} . (r_1^2 - r_0^2). \qquad (810)$$

134. *Corollary.* When the surface is curved, let γ denote the inclination of the tangent plane to the plane of $x\,y$, and,

Area of a curved surface.

since the projection of a surface is equal to the product of the surface by the cosine of its inclination to its projection, (806) gives

$$\sigma = \int_x \int_y \,.\, \text{sec.}\ \gamma. \tag{811}$$

Hence, by (600) of vol. 1, where

$$V = 0 \tag{812}$$

is the equation of the surface,

$$\sigma = \int_x \int_y \,.\, \frac{\surd\,(D_x\,V^2 + D_y\,V^2 + D_z\,V^2)}{D_z\,V}$$

$$= \int_x \int_y \,.\, \surd\,(D_x\,z^2 + D_y\,z^2 + 1). \tag{813}$$

135. *Corollary.* When the surface is developable, it may be supposed to be developed into a plane, and its area found as that of a plane surface; or it must give the same result to refer the surface to axes, drawn upon it in such a way, that they would be straight lines when the surface was developed, and the rectangular coördinates would then be the length of the shortest lines, which would be drawn upon the surface to two of these axes, which would be perpendicular to each other.

136. *Corollary.* When the surface is one of revolution, the notation of § 129 gives, by § 134,

$$\sigma = \int \varphi \int_u \,.\, u\,\surd\,(D_u\,z^2 + 1)\,; \tag{814}$$

and if s denotes the arc of the generating curve,

$$\sigma = \int \varphi \int_u \,.\, u\,D_u\,s = \int \varphi \int_x \,.\, u\,D_x\,.\,s = \int \varphi \int_s \,.\, u. \tag{815}$$

137. *Corollary.* When the surface of revolution is included between four curves, of which two are the intersections with the surface of two planes which are perpendicular to the axis

of revolution, and the other two are the intersections with the surface of the planes, which may be called meridian planes, because they include the axis of revolution, and which are inclined to each other by an angle φ_2, (815) gives

$$\sigma = \varphi_2 \int_{u_0}^{u_1} . u \, D_u \, s = \varphi_2 \int_{z_0}^{z_1} . u \, D_z \, . \, s = \varphi_2 \int_{s_0}^{s_1} . u. \quad (816)$$

138. *Corollary.* If another surface of revolution were generated by the revolution of the arc in the preceding section, about an axis at the distance b from the former axis, and farther from the arc, so that for this new axis we have

$$u' = u + b. \quad (817)$$

(816) gives the value of the corresponding surface

$$\begin{aligned} \sigma' &= \varphi_2 \int_{u_0}^{u_1} (u \, D_u \, s + b \, D_u \, s) \\ &= \varphi_2 \int_{u_0}^{u_1} u \, D_u \, s + b \, \varphi_2 \, (s_1 - s_0) \\ &= \sigma + b \, \varphi_2 \, (s_1 - s_0). \end{aligned} \quad (818)$$

139. *Corollary.* Had the second axis been upon the opposite side of the arc, we should have had

$$u'' = b - u \quad (819)$$

$$\sigma'' = b \, \varphi_2 \, (s_1 - s_0) - \sigma. \quad (820)$$

140. *Corollary.* A curve *A B A′ B′* (fig. 8) is said to have a *centre* O when there is such a point that any chord, such as *A A′*, *B B′*, &c. which passes through it, is bisected by it; and such a chord is called a diam-

eter. The surface generated by the revolution of such a curve about an axis $C\,C'$ which does not intersect the curve, is called *an angular surface*, or, simply, *a ring*.

The notation

$$\left.\begin{array}{l} S = \text{the perimeter of the generating curve } A\,B\,D\,A'\,D'\,A, \\ \sigma = \text{the surface which would be generated by the revolution of } D\,B\,A\,D' \text{ about the diameter } D\,D' \text{ parallel to } CC', \\ b = \text{the distance of the axis } C\,C' \text{ from the centre,} \end{array}\right\} \quad (821)$$

gives by (818 and 820) for the whole surface of the ring,

$$\begin{aligned} \Sigma &= \sigma + \tfrac{1}{2}\,2\,b\,\pi\,S + \tfrac{1}{2}\,2\,b\,\pi\,S - \sigma \\ &= 2\,b\,\pi\,S. \end{aligned} \qquad (822)$$

141. ***Problem.*** *To transform the differential coefficient of a surface from one system of variables to another.*

Solution. Let l and m be the given variables, and let the second member of (803) be denoted by H, that is,

$$D^2_{l,\,m}\,.\,\sigma = H \qquad (823)$$

If, then, only one of the variables m is to be changed, and t is to be introduced instead of it by means of the equation

$$M = m, \qquad (824)$$

in which M is a given function of l and t; we have

$$\begin{aligned} D^2_{l\,.\,t}\,.\,\sigma &= D_t\,D_l\,.\,\sigma = D_m\,D_l\,.\,\sigma\,.\,D_t\,.\,m \\ &= D^2_{l,\,m}\,.\,\sigma\,.\,D_t\,M = H\,D_t\,.\,M. \end{aligned} \qquad (825)$$

Transformation of differential coefficient.

If the other variable l is also to be changed, and u to be introduced instead of it, by means of the equation

$$l = L, \tag{826}$$

in which L is a given function of t and u; we have

$$D^2_{t.u}\,\sigma = H\,D_t\,.\,M\,.\,D_u\,L. \tag{827}$$

142. *Corollary.* If M, in equation (824), instead of being a given function of t and l, were a given function of t and u, u might be eliminated by means of (826). It is more convenient, however, to eliminate its differential coefficient only from $D_t\,.\,M$, after having determined this differential coefficient by means of (826). Thus the differential of (826) relative to t is, by regarding u as a function of t,

$$0 = D_t\,L + D_u\,L\,.\,D_t\,u, \tag{828}$$

whence

$$D_t\,u = -\frac{D_t\,L}{D_u\,L}, \tag{829}$$

and (824) gives

$$\begin{aligned} D_t\,m &= D_t\,M + D_u\,M\,.\,D_t\,u \\ &= \frac{D_t\,M\,.\,D_u\,L - D_u\,M\,.\,D_t\,L}{D_u\,L}. \end{aligned} \tag{830}$$

But m is obviously to be substituted for M in (827), whence we have by (827 and 830),

$$D^2_{t.u}\,\sigma = H\,(D_t\,M\,.\,D_u\,L - D_u\,M\,.\,D_t\,L). \tag{831}$$

143. *Corollary.* The two preceding articles may be applied to the transformation of any second differential coefficient of two successive variables.

144. Examples.

1. To find the area of the segment of an ellipse included between two parallel lines.

Transformation of differential coefficient.

Solution. Let the ellipse be referred to conjugate axes, as in (74) of vol. 1, in which the axis of y is drawn parallel to the given lines; and (804 and 807) give, since in this case

$$y_0 = -y_1 \tag{832}$$

is the ordinate y of the ellipse, if α is the angle of the axes

$$\sigma = 2 \int_{x_0}^{x_1} . \, y \sin. \alpha. \tag{833}$$

If, now, we take θ so that

$$x = A \cos. \theta, \tag{834}$$

we have

$$y = B \sin. \theta, \tag{835}$$

$$D_\theta . x = - A \sin. \theta, \tag{836}$$

$$\sigma = 2 \sin. \alpha \int_{\theta_1}^{\theta_0} A \, B \sin.^2 \theta = A \, B \sin. \alpha \int_{\theta_1}^{\theta_0} (1 - \cos. 2\,\theta)$$

$$= A \, B \sin. \alpha \, [\theta_0 - \theta_1 - \tfrac{1}{2} (\sin. 2\,\theta_0 - \sin. 2\,\theta_1)] \tag{837}$$

$= \frac{B}{A} \sin. \alpha$ (corresponding area of a segment of a circle whose radius is A).

2. To find the area of a sector of an ellipse, when the vertex of the sector is at the centre of the ellipse.

Solution. In this case (834 and 835) give, when A and B are the semiaxes,

$$r \cos. \varphi = A \cos. \theta, \qquad r \sin. \varphi = B \sin. \theta, \tag{838}$$

$$\tan. \varphi = \frac{B}{A} \tan. \theta, \qquad D_\theta . \varphi = \frac{B}{A} \frac{\cos.^2 \varphi}{\cos.^2 \theta}, \tag{839}$$

$$r^2 D_\theta . \varphi = A \, B; \tag{840}$$

whence, by (810), putting zero for r_0,

$$\sigma = \tfrac{1}{2} A B \, (\theta_1 - \theta_0). \tag{841}$$

Area of elliptic segment and sector.

Corollary. The whole area of the ellipse is $\pi A B$. (842)

3. To find the area of a sector of an ellipse, when the vertex of the sector is at a focus.

Solution. If the origin of coördinates is at the focus, (834 and 835) give

$$r \cos. \varphi = x = A \cos. \theta - A e = A (\cos. \theta - e) \tag{843}$$

$$r \sin. \varphi = y = B \sin. \theta \tag{844}$$

$$\tan. \varphi = \frac{B}{A} \cdot \frac{\sin. \theta}{\cos. \theta - e} \tag{845}$$

$$D_\theta \varphi = \frac{B}{A} \cdot \frac{(1 - e \cos. \theta) \cos.^2 \varphi}{(\cos. \theta - e)^2} \tag{846}$$

$$r^2 D_\theta \varphi = A B . (1 - e \cos. \theta), \tag{847}$$

whence, by (810),

$$\sigma = A B . [\theta_1 - \theta_0 - e (\sin. \theta_1 - \sin. \theta_0)]. \tag{848}$$

4. To find the area of the hyperbolic segment included between two parallel lines.

Ans. If the hyperbola is referred to conjugate axes as in (90) of vol. 1, in which the axis of y is parallel to the given lines, if γ is the angle of the axes, and if θ is taken so that

$$x = A \operatorname{Cos.} \theta, \quad y = B \operatorname{Sin.} \theta, \tag{849}$$

the area is

$$\sigma = \tfrac{1}{2} A B \sin. \gamma (\operatorname{Sin.} 2 \theta_1 - \operatorname{Sin.} 2 \theta_0 + 2 \theta_1 - 2 \theta_0). \tag{850}$$

5. To find the area of the hyperbolic sector, the vertex of which is at the centre of the hyperbola.

Ans. With the notation of the preceding example, the area is

$$\sigma = A\,B\,(\theta_1 - \theta_0), \qquad (851)$$

in which A and B are the semi-axes.

6. To find the area of the hyperbolic sector, the vertex of which is at one of the foci.

Ans. With the notation of the preceding example, the area is

$$\sigma = A\,B\,[(\theta_1 - \theta_0) - e\,(\text{Sin. }\theta_1 - \text{Sin. }\theta_0)]. \qquad (852)$$

7. To find the hyperbolic segment included between an asymptote, the curve, and two straight lines drawn parallel to the other asymptote.

Solution. It is convenient, in this case, to take the two asymptotes for the oblique axes, for which α and β in (86) of vol. I. must have the values

$$\tan.\,\alpha = \frac{B}{A}, \qquad \tan.\,\beta = -\frac{B}{A}; \qquad (853)$$

whence (86) gives for the equation of the hyperbola, referred to its asymptotes,

$$x\,y = \tfrac{1}{4}\,(A^2 + B^2). \qquad (854)$$

The area of the required segment is, then, by (807, 853 and 854), if the axis of y is the asymptote parallel to the given lines,

$$\sigma = \sin.\,2\,\alpha \int_{x_0}^{x_1} y = \frac{2\,A\,B}{A^2 + B^2} \int_{x_0}^{x_1} y$$

$$= \tfrac{1}{2}\,A\,B \int_{x_0}^{x^1} \frac{1}{x}$$

$$= \tfrac{1}{2}\,A\,B \log.\,\frac{x_1}{x_0}. \qquad (855)$$

Area of parabolic and cycloidal segments.

8. To find the area of the parabolic segment included between two parallel lines.

Ans. If the parabola is referred to oblique axes as in (100) of vol. 1, of which the axis of y is parallel to the given lines, and if α is the angle of the two axes, the area is

$$\sigma = \tfrac{4}{3} (y_1\ x_1 - y_0\ x_0). \tag{856}$$

9. To find the area of the parabolic sector, of which the vertex is at the focus.

Ans. If P' is the distance from the vertex to the focus, if the origin is at the focus and the angle φ counted from the vertex, the area is

$$\sigma = 2\,P\,(\tan.\ \tfrac{1}{2}\ \varphi_1 - \tan.\ \tfrac{1}{2}\ \varphi_0). \tag{857}$$

10. To find the area of the segment included between the curve, the axis of x, and two lines drawn parallel to the axis of y, of the curve known as the parabola of the order a, which has for its equation

$$y = A\,x^a. \tag{858}$$

$$\textit{Ans.}\quad \sigma = \frac{x_1\ y_1 - x_0\ y_0}{a+1}. \tag{859}$$

11. To find the are of the segment of a cycloid given by equations (130 and 131) of vol. 1, included between the curve, the axis of x, and two lines drawn parallel to the axis of y.

$$\textit{Ans.}\ \sigma = R^2[\tfrac{3}{2}(\theta_1 - \theta_0) - 2(\sin.\theta_1 - \sin.\theta_0) + \tfrac{1}{4}(\sin.2\,\theta_1 - \sin.2\,\theta_0)]. \tag{860}$$

Corollary. The whole area included between a branch of the cycloid and the axis of x, is

$$\sigma = 3\,\pi\,R^2, \tag{861}$$

$=$ three times the area of the generating circle.

12. To find the area of the segment of a cycloid, which is included between the curve and a line drawn parallel to the axis of y.

Ans. $\sigma = R^2 [(\pi - \theta)(1 + 2 \cos. \theta) - 2 \sin. \theta - \frac{1}{2} \sin. 2\theta]$.

13. To find the area of a sector of the spiral of equation (133) of vol. 1, when the vertex of the sector is at the origin.

$$Ans. \quad \sigma = \frac{\frac{1}{2} R^2}{2n+1} (\varphi_1{}^{2n+1} - \varphi_0{}^{2n+1}). \tag{863}$$

14. To find the area of a sector of the hyperbolic spiral, the equation of which is (135) of vol. 1, when the vertex of the sector is at the origin.

$$Ans. \quad \sigma = 2\pi^2 R^2 \left(\frac{1}{\varphi_0} - \frac{1}{\varphi_1}\right). \tag{864}$$

15. To find the area of a sector of the logarithmic spiral, of which the equation is

$$r = a e^{\varphi}, \tag{865}$$

when the vertex of the sector is at the origin.

$$Ans. \quad \sigma = \tfrac{1}{4} a^2 (e^{2\varphi_1} - e^{2\varphi_0}). \tag{866}$$

16. Given the area σ of a surface included between any lines whatever, the combination of which considered as one line which in general is discontinuous, is represented by the equation

$$F.(x, y) = 0, \tag{867}$$

to find the area σ' of the surface bounded by the line or system of lines

$$F\left(\frac{x}{a}, \frac{y}{b}\right) = 0. \tag{868}$$

$$Ans. \quad \sigma' = a\,b\,\sigma. \tag{869}$$

Area of a zone of an ellipsoid.

Corollary. If a and b are equal, the surfaces are similar, and (869) gives

$$\sigma' = a^2 \sigma ; \tag{870}$$

that is, *the areas of similar surfaces areproportional to the squares of their dimensions.*

17. To find the area of the zone of an oblate ellipsoid of revolution which is included between two planes drawn perpendicular to the axis of revolution.

Solution. Let the notation be that of Example 1, of § 130, and (816) gives, for the area,

$$\sigma = 2\pi \int_{x_0}^{x_1} x\, D_x \,.\, s$$

$$= 2\pi A^2 \int_{\theta_0}^{\theta_1} \sin.\theta \,.\, \Delta\theta = 2\pi A \int_{\theta_0}^{\theta_1} \sin.\theta \sqrt{[B^2 + (A^2 - B^2)\cos.^2\theta]}. \tag{871}$$

Let the angle ω be so taken that

$$B \text{ Sin. } \omega = A\, e \cos.\theta ; \tag{872}$$

and we shall have

$$-A\, e \sin.\theta\, D_\omega \,.\, \theta = B \text{ Cos. } \omega ; \tag{873}$$

whence

$$\sigma = \frac{2\pi B^2}{e} \int_{\omega_1}^{\omega_0} .\, \text{Cos.}^2\, \omega$$

$$= \frac{\pi B^2}{e} \int_{\omega_1}^{\omega_0} .\, (1 + \text{Cos. } 2\,\omega)$$

$$= \frac{\pi B^2}{e} \left[(\omega_0 - \omega_1) + \tfrac{1}{2} (\text{Sin. } 2\,\omega_0 - \text{Sin. } 2\,\omega_1)\right]. \tag{874}$$

18. To find the area of the zone of a prolate ellipsoid of revolution which is included between two planes drawn perpendicular to the axis of revolution.

Ans. With the notation of Example 1, of § 130, and putting

$$\cos. \varpi = e \cos. \theta, \tag{875}$$

the area is

$$\sigma = \frac{\pi A B}{e} [(\varpi_1 - \varpi_0) + \tfrac{1}{2} (\sin. 2 \varpi_1 - \sin. 2 \varpi_0)]. \tag{876}$$

19. To find the area of the zone of the hyperboloid of revolution formed by the revolution of an arc of an hyperbola about the transverse axis.

Ans. If the equations of the generating hyperbola are

$$x = A \text{ Cos. } \theta, \qquad y = B \text{ Sin. } \theta, \tag{877}$$

and if ϖ is taken so that

$$e \text{ Cos. } \theta = \sec. \varpi, \tag{878}$$

the area is

$$\sigma = \frac{\pi A B}{e} \left[\frac{\sin. \varpi_1}{\cos.^2 \varpi_1} - \frac{\sin. \varpi_0}{\cos.^2 \varpi_0} + \log. \frac{\text{tang.} (45° + \frac{1}{2} \varpi_1)}{\text{tang.} (45° + \frac{1}{2} \varpi_0)} \right]. \tag{879}$$

20. To find the area of the zone of the paraboloid of revolution, included between two planes, which are perpendicular to the axis of revolution.

Ans. If P is the distance from the vertex to the focus, and if θ is so taken that

$$y = 2 P \tan. \theta, \tag{880}$$

the area is

$$\sigma = \tfrac{8}{3} \pi P^2 (\sec.^3 \theta_1 - \sec.^3 \theta_0). \tag{881}$$

21. To find the area of the zone generated by the revolution of an arc of a parabola about the axis of y of the preceding example.

Ans. If θ is taken so that

$$x + P = P \sec. \theta, \tag{882}$$

and if σ' is the value of σ in (879),

the area is
$$\sigma = \frac{P^2 e}{A B} \sigma'. \tag{883}$$

22. To find the area of the zone generated by the revolution of an arc of a cycloid about the axis of x in (130) of vol. 1. The arc is supposed to commence with θ.

Ans. With the notation of equations (130 and 131) of vol. 1, the area is

$$\sigma = 16 \pi R^2 (\tfrac{2}{3} - \tfrac{2}{3} \cos. \tfrac{1}{2} \theta - \tfrac{1}{3} \sin.^2 \tfrac{1}{2} \theta . \cos. \tfrac{1}{2} \theta). \tag{884}$$

23. To find the area of the zone generated by the revolution of an arc of a cycloid about the axis of y in (131) of vol. 1. The arc is supposed to commence with θ.

Ans. With the notation of the preceding example, the area is

$$\sigma = 16 \pi R^2 (\sin. \tfrac{1}{2} \theta - \tfrac{1}{2} \theta \cos. \tfrac{1}{2} \theta - \tfrac{1}{3} \sin.^3 \tfrac{1}{2} \theta). \tag{885}$$

145. *Problem. To find the area of the zone generated by the revolution of a given arc of a plane curve about an axis in the same plane with the arc, when the areas of the two zones are known which are generated by the revolution of the arc about two axes in the plane, which are perpendicular to each other.*

Solution. Let the two perpendicular axes be those of x and y, and let the given areas be, by (816),

$$\sigma' = 2\,\pi \int_{s_0}^{s_1} .\, y, \tag{886}$$

$$\sigma'' = 2\,\pi \int_{s_0}^{s_1} .\, x. \tag{887}$$

Let the new axis be inclined to the axis of x by an angle α, and pass at a distance a from the origin, and the required area is

$$\begin{aligned}\sigma &= \pm\, 2\,\pi \int_{s_0}^{s_1} .\, (y \cos.\, \alpha - x \sin.\, \alpha - a) \\ &= \pm\, 2\,\pi\, [\sigma' \cos.\, \alpha - \sigma'' \sin.\, \alpha - a\,(s_1 - s_0)],\end{aligned} \tag{888}$$

in which that sign is to be adopted which renders the second member positive.

146. *Problem. To draw the curve line subject to given conditions, which includes a maximum or minimum surface.*

Solution. This problem, like that of § 116, involves the maximum or minimum of a definite integral, and is therefore solved in a similar way, by the method of variations. There is, in this case, however, a double integral, and the first integral refers evidently not to disconnected points, but to the bounding lines of the surface, so that the determination of these lines may involve the method of variations, even when the general form of the surface is given. The determination of the form of the surface will admit of more lucid discussion in a chapter upon the curvature of surfaces, and the present chapter will be confined to the consideration of the bounding line.

The equation of the surface being given, the form of its second differential coefficient is known, and is independent of

the limiting lines, so that an integration can be directly performed, and the required integral be reduced to the form (756), and the process of finding the maximum or minimum becomes identical with that of § 116.

147. *Corollary.* A kind of equation of condition is often connected with this problem, wholly different from those referred to in § 116. Each of the equations (754) is an equation which is satisfied by the coördinates of each point of the required curve, and is thus equivalent to an infinite number of equations. But an equation, of the class here alluded to, is a single equation, involving the coördinates of every point of the curve. An instance of such an equation is the one which expresses that the bounding curve must be of a given length, or that the definite integral (741) must have a given value.

All equations of this kind would appear to depend, necessarily, upon definite integrals, and they may be introduced into the equation of maximum or minimum for the purpose of elimination by the method of § 119. It must be observed, however, that *the multipliers* λ, μ, &c, *of these equations are always constant.* For each of these equations does not determine any relation between δx, δy, &c. which is applicable to each point of the curve, but only a particular relation by which one of the variations, as δx, may be determined for one of the points in terms of the values of the variations for all the points. The corresponding multiplier λ, therefore, must have that particular value which shall cause this single value of δx to disappear from the equation; that is, λ must be constant.

148. Examples.

1. To find the plane curve which, having a given length, encloses the maximum area.

Solution. The function to be a maximum is, by (806),

$$\int_{x_0}^{x_1} y, \tag{889}$$

and the function (566) is to be constant. Hence if A is the constant multiplier introduced for the purpose of elimination, the equation is, by the reduction of § 121,

$$1 - A\, D_x \left(\frac{D\,y}{D\,s}\right) = 0, \tag{890}$$

or by the notation of § 148 of B. II., and by (577 and 609 of vol. 1,

$$0 = 1 + A\, D_x \,.\, \cos.\, \nu \tag{891}$$

$$0 = D_\nu\, x + A\, D_\nu \cos.\, \nu$$

$$= \sin.\, \nu\; D_\nu\, s - A \sin.\, \nu \tag{892}$$

$$A = D_\nu\, s = \varrho\,; \tag{893}$$

that is, the curvature is constant, which is the property of no other curve than the circle; the required curve is, therefore, a circle; which has, already, been proved in the Elements of Geometry.

2. To find the plane curve which, being drawn from one given point to another given point, and having a given length, encloses the maximum area between the curve itself, its two extreme radii of curvature and its evolute.

Solution. By adopting the notation of the preceding article, the required area may be expressed in the form

$$\int_{\nu_0}^{\nu_1} .\, \varrho^2\,; \tag{894}$$

that of the arc will be

$$s = \int_{\nu_0}^{\nu_1} .\, \varrho. \tag{895}$$

Greatest or least surface.

Equations (576, 577 and 609) of vol. 1, give

$$D_\nu x = \sin.\,\nu\, D_\nu s = \varrho \sin.\,\nu, \tag{896}$$

$$D_\nu y = -\cos.\,\nu\, D_\nu s = -\varrho \cos.\,\nu. \tag{897}$$

The given differences of the coördinates of the extreme points of the curve are, then,

$$x_1 - x_0 = \int_{\nu_0}^{\nu_1} .\, \varrho \sin.\,\nu, \tag{898}$$

$$y_1 - y_0 = -\int_{\nu_0}^{\nu_1} .\, \varrho \cos.\,\nu. \tag{899}$$

If, therefore, A, B, C are the constant multipliers of (895, 898 and 899), introduced for the purpose of elimination, the equation of the maximum or minimum is

$$2\varrho + A + B \sin.\,\nu - C \cos.\,\nu = 0. \tag{900}$$

Let H and α be taken so that

$$B = H \cos.\,\alpha, \qquad C = H \sin.\,\alpha; \tag{901}$$

and (900) becomes

$$2\varrho + A + H \sin.\,(\nu - \alpha) = 0; \tag{902}$$

and by putting

$$\nu' = \nu - \alpha, \tag{903}$$

$$2\varrho + A + H \sin.\,\nu' = 0; \tag{904}$$

which shows that (900) may be reduced to the form (904), from which the term containing $\cos.\,\nu$ disappears, by merely changing the direction of the axis of x. It does not, then, diminish the generality of the solution to put

$$C = 0; \tag{905}$$

by which (900) becomes

$$2\varrho + A + B \sin.\,\nu = 0. \tag{906}$$

The curve is easily expressed in rectangular coördinates by the equations

$$x = \tfrac{1}{2} A \cos. \nu + \tfrac{1}{8} B \sin. 2\nu + \tfrac{1}{8} \nu, \tag{907}$$

$$y = \tfrac{1}{2} A \sin. \nu - \tfrac{1}{8} B \cos. 2\nu. \tag{908}$$

Corollary. When the extreme points are not fixed, the equation (900) becomes

$$2\varrho + A = 0; \tag{909}$$

that is, the curve is *a circle.*

Corollary. When the length of the curve is not given, the equation (906) becomes

$$2\varrho + B \sin. \nu = 0; \tag{910}$$

which is, evidently, from example 3 of § 151 of B. II., *a cycloid.*

CHAPTER VIII.

THE CURVATURE OF SURFACES.

149. ***Problem.*** *To find the curvature of a given surface at any point in any direction.*

Solution. Let the tangent plane to the surface at any one of its points be taken for the plane of the coördinates x and y, so that the normal may be the axis of z. We have, then, at this point,

$$D_x z = 0, \qquad D_y z = 0 ; \tag{911}$$

and if ϱ_x and ϱ_y are the radii of curvature at the point of the intersections of the planes of xz and yz with the surface, equation (610) of vol. 1 gives

$$\frac{1}{\varrho_x} = D_x^2 z, \qquad \frac{1}{\varrho_y} = D_y^2 z. \tag{912}$$

The radius of curvature ϱ of a section made in any intermediate direction by a normal plane, which is inclined to the axis of x by the angle α, is derived from the equation

$$\frac{1}{\varrho} = D_u^2 z, \tag{913}$$

if u denotes the distance of a point of the curve of intersection from the axis of z. But the coördinates of one of these points are

$$x = u \cos. \alpha, \quad y = u \sin. \alpha ; \tag{914}$$

whence, in general,

$$D_u\, z = \cos.\, \alpha\, D_x\, z + \sin.\, \alpha\, D_y\, z, \tag{915}$$

$$\frac{1}{\varrho} = D_u^2\, z = \cos.^2 \alpha\, D_x^2\, z + 2 \sin.\, \alpha \cos.\, \alpha\, D_{x.y}^2\, z + \sin.^2 \alpha\, D_y^2\, z$$

$$= \frac{\cos.^2 \alpha}{\varrho_x} + \frac{\sin.^2 \alpha}{\varrho_y} + 2 \sin.\, \alpha \cos.\, \alpha\, D_{x.y}^2\, z. \tag{916}$$

150. *Corollary.* The radius of curvature ϱ', in a direction perpendicular to that of ϱ, is given by the equation

$$\frac{1}{\varrho'} = \frac{\sin.^2 \alpha}{\varrho_x} + \frac{\cos.^2 \alpha}{\varrho_y} - 2 \sin.\, \alpha \cos.\, \alpha\, D_{x.y}^2\, z. \tag{917}$$

151. *Corollary.* The sum of (916 and 917) is

$$\frac{1}{\varrho} + \frac{1}{\varrho'} = \frac{1}{\varrho_x} + \frac{1}{\varrho_y}\,; \tag{918}$$

that is, *the sums of the reciprocals of the two radii of curvature of any two perpendicular sections at a given point of a surface is a constant quantity.*

152. *Corollary.* If ϱ were the maximum radius of curvature at the point, ϱ' would obviously be the minimum radius of curvature; whence

The directions of greatest and least curvature of a surface at any point are perpendicular to each other.

153. *Corollary.* The differenee between (916 and 917) is

$$\frac{1}{\varrho'} - \frac{1}{\varrho} = \cos.\, 2\alpha \left(\frac{1}{\varrho_y} - \frac{1}{\varrho_x}\right) - 2 \sin.\, 2\alpha\, D_{x.y}^2\, z, \tag{919}$$

and in the hypothesis of the preceding corollary, the first mem-

ber of (919) is a maximum. The differential coefficient of the second member, taken with reference to α, must be equal to zero, that is,

$$0 = \sin.\, 2\,\alpha \left(\frac{1}{\varrho_y} - \frac{1}{\varrho_x}\right) + 2 \cos.\, 2\,\alpha\, D^2{}_{x\,.\,y}\, z. \qquad (920)$$

The sum of (919) multiplied by $\cos.\, 2\,\alpha$, and of (920) by $\sin.\, 2\,\alpha$, is

$$\cos.\, 2\,\alpha \left(\frac{1}{\varrho'} - \frac{1}{\varrho}\right) = \frac{1}{\varrho_y} - \frac{1}{\varrho_x}. \qquad (921)$$

Hence, from (918),

$$\frac{1}{\varrho_x} = \tfrac{1}{2}\left(\frac{1}{\varrho'} + \frac{1}{\varrho}\right) - \tfrac{1}{2} \cos.\, 2\,\alpha \left(\frac{1}{\varrho'} - \frac{1}{\varrho}\right)$$

$$= \frac{\cos.^2\, \alpha}{\varrho} + \frac{\sin.^2\, \alpha}{\varrho'}, \qquad (922)$$

from which *the curvature of the surface can be found in a direction inclined by the angle α to the direction of maximum curvature.*

154. *Corollary.* One half of the difference between (919) multiplied by $\sin.\, 2\,\alpha$, and (920) by $\cos.\, 2\,\alpha$, is

$$D^2{}_{x\,.\,y}\, z = -\tfrac{1}{2} \sin.\, 2\,\alpha \left(\frac{1}{\varrho'} - \frac{1}{\varrho}\right). \qquad (923)$$

155. *Corollary.* For the direction of the maximum or minimum, α is zero or a right angle, and, therefore, for either of these directions,

$$D^2{}_{x\,.\,y}\, z = 0\,; \qquad (924)$$

that is, *with a small motion of the point of contact in the direction of the greatest or least curvature, the tangent plane rotates about a line perpendicular to the direction of the motion of the point.*

Direction of no curvature.

156. *Corollary.* When α is half a right angle, (921 and 922) give

$$\varrho_y = \varrho_x, \tag{925}$$

$$\frac{1}{\varrho_y} = \frac{1}{\varrho_x} = \tfrac{1}{2}\left(\frac{1}{\varrho} + \frac{1}{\varrho'}\right). \tag{926}$$

157. *Corollary.* When the values of ϱ and ϱ' have opposite signs, neither of the corresponding curvatures is strictly a minimum, but the two curvatures are *the greatest curvatures in opposite directions.* There are, in this case, two intermediate directions of no curvature, corresponding by (922) to the values of α,

$$\text{tang. } \alpha = \pm \sqrt{\left(-\frac{\varrho'}{\varrho}\right)}. \tag{927}$$

The sections of the surface, made in these directions, have a contact of the second order with the tangent plane, and correspond, in general, to points of contrary flexure.

158. *Corollary.* In the case of a point of contact for which the greatest and least curvatures are in opposite directions and equal, we have

$$\varrho = -\varrho'; \tag{928}$$

whence, by (918),

$$\varrho_x = -\varrho_y; \tag{929}$$

that is, the curvatures in any two directions, which are perpendicular to each other, are equal and opposite.

We have also in this case, by (927),

$$\alpha = \pm 45^\circ \tag{930}$$

for the angles, which the directions of no curvature make with the direction of greatest curvature.

159. *Corollary*. If the curvature were required of a section, the plane of which did not include the normal, it might be found by referring the surface to an oblique system of coordinates, of which the tangent plane was the plane of $x\,y$, the cutting plane that of $x\,z'$, the axis of x being the intersection of these two planes, and the axes of y and z' being perpendicular to that of x. This system might be obtained from the rectangular one, which has the same axes of x and y, but in which the axis of z is the normal, by putting

$\theta =$ the inclination of the axis of z to that of z', (931)

$=$ the complement of the inclination of the given plane to the tangent plane,

which gives

$$z = z' \cos.\,\theta, \tag{932}$$

$$D_x^2\, z = D_x^2\, z' \cos.\,\theta\,; \tag{933}$$

or, by putting $\varrho_\theta =$ the radius of curvature of the inclined section

$$\frac{1}{\varrho} = \frac{1}{\varrho_\theta} \cos.\,\theta, \tag{934}$$

$$\varrho_\theta = \varrho \cos.\,\theta. \tag{935}$$

160. *Corollary*. If the axes, in the preceding corollary, were rectangular, that of y being perpendicular to the given plane, and those of x and z situated in any way whatever in that plane, equation (610) of vol. 1 gives

$$\frac{1}{\varrho_\theta} = \frac{D_x^2\, z}{[1 + (D_x\, z)^2]^{\frac{3}{2}}}. \tag{936}$$

If we put

$$\left.\begin{aligned} \tau &= \text{the angle of } \varrho_\theta \text{ and } z \\ \gamma &= \text{the angle of } \varrho \text{ and } z, \end{aligned}\right\} \tag{937}$$

and observe that the plane of ϱ and ϱ_θ is perpendicular to that

of z and x, so that if a sphere were described with the point of contact for the centre, the arcs θ, τ, γ would form upon the surface a right triangle, of which γ was the hypothenuse, we have

$$\cos. \gamma = \cot. \tau \cos. \theta. \tag{938}$$

But the comparison of (811 and 813) gives

$$\sec. \gamma = \sqrt{[1 + (D_x z)^2 + (D_y z)^2]}, \tag{939}$$

and we have, obviously,

$$\sec. \tau = \sqrt{[1 + (D_x z)^2]}; \tag{940}$$

whence

$$\frac{1}{\varrho} = \frac{1}{\varrho_\theta} \cdot \frac{\sec. \tau}{\sec. \gamma} = \frac{D_x^2 . z}{1+(D_x z)^2} \cdot \cos. \gamma$$

$$= \frac{D_x^2 . z}{1 + (D_x z)^2} \cdot \frac{1}{\sqrt{[1 + (D_y z)^2 + (D_x z)^2]}}. \tag{941}$$

161. *Corollary.* The curvature of a section of the surface made by a plane which includes the axis of z, and is inclined to the plane of $z\,x$ by the angle ϵ, may be found by the formula

$$\frac{1}{\varrho} = \frac{D_u^2 z}{1+(D_u z)^2} \cdot \cos. \gamma, \tag{942}$$

in which $u =$ the distance of any point of the section from the axis of z,

whence

$$x = u \cos. \epsilon, \quad y = u \sin. \epsilon; \tag{943}$$

$$D_u x = \cos. \epsilon, \quad D_u y = \sin. \epsilon; \tag{944}$$

$$D_u z = \cos. \epsilon . D_x z + \sin. \epsilon\, D_y z, \tag{945}$$

$$D_u^2 z = \cos.^2 \epsilon . D_x^2 z + 2 \sin. \epsilon \cos. \epsilon\, D_{x.y}^2 z + \sin.^2 \epsilon . D_y^2 z; \tag{946}$$

$$\frac{1}{\varrho} = \frac{(D_x^2 z + 2 \tan. \epsilon . D_{x.y}^2 z + \tan.^2 \epsilon . D_y^2 z) \cos. \gamma}{1 + D_x z^2 + 2 \tan. \epsilon\, D_x z\, D_y z + (1 + D_y z^2) \tan.^2 \epsilon}; \tag{947}$$

Curvature of any part of the surface.

and since the coördinates x, y, z do not themselves occur in this value of the reciprocal of the radius of curvature, but only their differentials, (947) is applicable to any point of the surface, and to any direction of the curvature, it being observed that ε is the angle, which the plane, drawn through the axis of z and parallel to this direction, makes with the plane of $x\,y$.

162. *Corollary.* When the plane which is parallel to the required direction of curvature is also parallel to the radius of curvature, (601, 598 and 599) of vol. 1 give

$$\tan. \varepsilon = \frac{\cos. \beta}{\cos. \alpha} = \frac{D_y z}{D_x z}; \tag{948}$$

whence the product of the denominator of (947), by $D_x z^2$, becomes

$$\begin{aligned} D_x z^2 + D_x z^4 + 2\, D_x z^2\, D_y z^2 + D_y z^2 + D_y z^4 \\ = (D_x z^2 + D_y z^2)\ (1 + D_x z^2 + D_y z^2) \\ = (D_x z^2 + D_y z^2) \sec.^2 \gamma\,; \end{aligned} \tag{949}$$

and (947) becomes

$$\frac{1}{\varrho} = \frac{D_x z^2\, D_x^2 z + 2\, D_x z\, D_y z\, D^2_{x\,.\,y} z + D_y z^2\, D_y^2 z}{D_x z^2 + D_y z^2} \cos.^3 \gamma. \tag{950}$$

163. *Corollary.* When the direction of curvature is perpendicular to that of the preceding article, the plane which is parallel to it is also perpendicular to that of the preceding article; whence, in this case,

$$\tan. \varepsilon' = -\cot. \varepsilon = -\frac{D_x z}{D_y z},$$

and (947) becomes

$$\frac{1}{\varrho'} = \frac{D_y z^2\, D_x^2 z - 2\, D_x z\, D_y z\, D^2_{x\,.\,y} z + D_x z^2\, D_y^2 z}{D_x z^2 + D_y z^2} \cos. \gamma. \tag{951}$$

Sum of any two perpendicular radii of curvature.

164. *Corollary.* The sum of (950 and 951) is

$$\frac{1}{\varrho}+\frac{1}{\varrho'}=\frac{(1+D_y z^2)D_x^2 z-2D_x z\, D_y z\, D^2_{x.y} z+(1+D_x z^2)D_y^2 z}{\text{sec.}^3\,\gamma}; \tag{952}$$

which is, by (918), *the sum of the reciprocals of the greatest and least radii of curvature at the point x, y, z; or it is the sum of any two perpendicular radii of curvature.*

165. *Problem. To find the greatest or least surface which can be drawn under given conditions.*

Solution. This form of statement embraces that portion of the problem of § 146 which was reserved for this chapter. Since a single equation between the coördinates of each point is sufficient to determine the surface, no such equation can be given; but there may be particular conditions involving definite integrals, like those referred to in § 146.

166. *Corollary.* When there is no condition whatever, the required surface is absolutely *the least surface of all which have the same boundary.*

In this case, the integral to be a minimum is (811 or 813), the variation of which gives

$$\textstyle\int_x\int_y .\ \text{cos.}_\gamma\,(D_x z\, D_x \delta z+D_y z\, D_y \delta z)=0. \tag{953}$$

But, by integration,

$$\textstyle\int_x\int_y .\ \text{cos.}_\gamma\, D_x z\, D_x\delta z=\int_y\int_x .\ \text{cos.}_\gamma\, D_x z\, D_x\delta z$$
$$\textstyle=\int_y .\ D_x z\ \text{cos.}_\gamma\, \delta z-\int_x\int_y .\ D_x(\text{cos.}_\gamma\, D_x z)\,\delta z, \tag{954}$$

$$\textstyle\int_x\int_y .\ \text{cos.}_\gamma\, D_y z D_y\delta z=\int_x .D_y z\ \text{cos.}_\gamma\,\delta z-\int_x\int_y .\ D_y(\text{cos.}_\gamma\, D_y z)\,\delta z; \tag{955}$$

whence, by regarding only the terms under the double sign of integration,

$$0 = D_x(\cos.\gamma\, D_x z) + D_y(\cos.\gamma\, D_y z)$$

$$= \cos.\gamma(D^2_x z + D^2_y z) + D_x z\, D_x . \cos.\gamma + D_y z\, D_y \cos.\gamma. \quad (956)$$

But

$$D_x . \cos.\gamma = D_x . (D_x z^2 + D_y z^2 + 1)^{-\frac{1}{2}}$$

$$= -\cos.^3\gamma\,(D_x z\, D^2_x z + D_y z\, D^2_{x.y} z), \quad (957)$$

$$D_y . \cos.\gamma = -\cos.^3\gamma\,(D_x z\, D^2_{x.y} z + D_y z\, D^2_y z); \quad (958)$$

which, substituted in (956), give by (952),

$$0 = \frac{(1+D_y z^2)\, D^2_x z - 2\, D_x z\, D_y z\, D^2_{x.y} z + (1+D_x z^2)\, D^2_y z}{\sec.^3\gamma}$$

$$= \frac{1}{\varrho} + \frac{1}{\varrho'} \quad (959)$$

or $\quad \varrho' = -\varrho; \quad (960)$

so that this surface is one in which every point is a case of § 158; that is, *in which the curvatures, in directions perpendicular to each other, are equal and opposite.*

The plane is the most simple instance of such a surface, but there are other examples to an unlimited extent.

167. *Corollary.* The complete determination of these surfaces must be reserved for a chapter upon the integration of partial differential equations; but the following ingenious construction, proposed by Monge, notwithstanding its obvious want of practical utility, which was acknowledged by its author, is

sufficient to exhibit the possibility of such a surface, and give some idea of its nature.

Let any curve line, of single or double curvature, be drawn at pleasure in space. Produce all its radii of curvature towards the opposite side of the curve from the centres of curvature, and to a distance from the curve exactly equal to the corresponding radii of curvature. The given curve line may, then, be assumed as *a line of curvature* of the required surface; that is, as a line which lies upon the surface and has at each point, the same curvature with the surface in the direction of this line. The produced radii of curvature, will be the radii of curvature of the surface in directions perpendicular to the given curve; and if the extremities of those produced radii, which are the corresponding centres of curvature, are fixed, and if all the points of the given curve are rotated with the radii about these centres, moving in planes perpendicular to the given line, each element of the given line will describe an element of the required surface. The given line in its new position will acquire a new form and become a new line of curvature, from which another elementary zone of the surface may be described by a repetition of the above process.

The small arc, through which each point of the curve must move, is not arbitrary, but is limited by the condition that two successive radii must be in the same plane, so as to meet at the centre of curvature.

168. *Corollary.* If the given curve of the preceding construction were a circle, the resulting surface would be a surface of revolution about an axis perpendicular to the plane of the circle and passing through its centre. The particular form

Minimum surface of revolution.

of this surface may be investigated by taking the axis of z for that of revolution, so that if

$$u = x^2 + y^2, \tag{961}$$

z will be a function of u, and will contain no other function of x and y. Hence

$$\left.\begin{array}{l} D_x z = D_u z \,.\, D_x u = 2\, x\, D_u z, \\ D_y z = 2\, y\, D_u z, \\ D_x^2 z = 2\, D_u z + 4\, x^2\, D_u^2 z, \quad D_{x.y}^2 z = 4\, x\, y\, D_u^2 z, \\ D_y^2 z = 2\, D_u z + 4\, y^2\, D_u^2 z\,; \end{array}\right\} \tag{962}$$

which, substituted in (958), give, by dividing by 4 cos.^{3} γ,

$$D_u z + 2\, u\, D_u z^3 + u\, D_u^2 z = 0. \tag{963}$$

By putting

$$v = \sqrt{u}, \tag{964}$$

we have

$$\left.\begin{array}{l} D_u z = \dfrac{1}{2\, v}\, D_v z, \\ D_u^2 z = -\dfrac{1}{4\, v^3}\, D_v z + \dfrac{1}{4\, v^2}\, D_v^2 z\,; \end{array}\right\} \tag{965}$$

which, substituted in (963), give

$$\frac{D_v z + D_v z^3}{v} + D_v^2 z = 0. \tag{966}$$

Hence

$$\begin{aligned} 0 &= \frac{1}{v} + \frac{D_v^2 z}{D_v z + D_v z^3} = 0 \\ &= \frac{1}{v} + \frac{D_v^2 z}{D_v z} - \frac{D_v z\, D_v^2 z}{1 + D_v z^2}\,; \end{aligned} \tag{967}$$

the integral of which is, by introducing A as an arbitrary constant,

$$\log.\, A = \log.\, v + \log.\, D_v z - \log.\, \sqrt{(1 + D_v z^2)}, \tag{968}$$

Minimum surface of revolution.

or
$$\frac{A}{v} = \frac{D_v z}{\sqrt{(1 + D_v z^2)}}. \qquad (969)$$

Hence
$$D_v z = \frac{A}{\sqrt{(v^2 - A^2)}}, \qquad (970)$$

and if φ is taken so that
$$v = A \text{ Cos. } \varphi, \qquad (971)$$

(970) gives
$$D_\varphi . z = D_v z \,.\, D_\varphi v = A \text{ Sin. } \varphi \, D_v z$$
$$= A \text{ Sin. } \varphi \,.\, \frac{A}{A \text{ Sin. } \varphi} = A, \qquad (972)$$
$$z = A \varphi; \qquad (973)$$

and the equation of the surface is
$$\sqrt{(x^2 + y^2)} = A \text{ Cos. } \frac{z}{A}$$
$$= \tfrac{1}{2} A \left(e^{\frac{z}{A}} + e^{-\frac{z}{A}}\right). \qquad (974)$$

CHAPTER IX.

THE CUBATURE OF SOLIDS.

169. *Problem. To find the measure of the volume of a given solid.*

Solution. Let the conditions of the bounding line be expressed by an equation between three variables, l, m, and n. Suppose two surfaces drawn infinitely near each other, in such a way that n is constant throughout their extent. If, then, V denotes the required volume, we have

$d_n V =$ the lamina included between these two surfaces.

If two other surfaces are drawn infinitely near each other, in such a way that m is constant through their extent, we have

$d_m d_n V =$ the small solid rod included between these four surfaces.

If two more surfaces are drawn infinitely near each other, in such a way that l is constant throughout their extent, we have

$d_l d_m d_n V =$ the infinitely small parallelopiped included between these six surfaces. (975)

If s' denotes an arc of the intersection of two surfaces for which m and n are constant, s'' an arc of the intersection of two surfaces for which l and n are constant, s''' an arc of the intersection of two surfaces for which l and m are constant;

and if a' is the inclination of s'' to s''' at the point of meeting, a'' that of s' to s''', and a''' that of s' to s''; and if b is the inclination of s''' to the surface which includes s' and s''; the sides of the small parallelopiped will be $d\,s'$, $d\,s''$, $d\,s'''$;

the face which includes $d\,s'$ and $d\,s'' = \sin a'''\, d\,s'\, d\,s''$

the distance of this from the opposite face $= \sin.\, b\, d\,s'''$;

whence

$$d_l\, d_m\, d_n\, V = \sin.\, a'''\, \sin.\, b\;\, d\,s'\;\, d\,s''\;\, d\,s'''. \tag{976}$$

But l is the only variable in s', m the only one in s'', and n the only one in s''', whence the accents may be neglected, by dividing by $d\,l\,.\,d\,m\,.\,d\,n$, and (976) gives

$$D^3_{l,\,m,\,n}\, V = D_l\,.\,D_m\,.\,D_n\, V = \sin.\, a'''\, \sin.\, b\; D_l\, s\,.\,D_m\, s\,.\,D_n\, s\,; \tag{977}$$

in which $D_l\, s$, $D_m\, s$, and $D_n\, s$ may be deduced from the general expression for the differential of an arc in space, by putting successively each pair of the quantities l, m and n, equal to zero. The value of V is, then, the third integral of (977).

170. *Corollary.* If one of the vertices of the parallelopiped is taken for the centre of a sphere, a', a'', a''' will form, by the intersection of the sides of the parallelopiped with the surface of the sphere, a spherical triangle; in which b will be the distance of a''' from the opposite vertex.

Hence, if A' is the angle opposite a', and if M is the ratio of the sines of the sides to the sines of the opposite angles, so that

$$M = \frac{\sin.\, A}{\sin.\, a'}, \tag{978}$$

we have

$$\sin.\, b = \sin.\, a''\, \sin.\, A' = M \sin.\, a'\, \sin.\, a''\,; \tag{979}$$

Cubature of solids of revolution.

and (977) becomes

$$D^3_{l.m.n.}V = M\sin. a' \sin. a'' \sin. a'''. D_l s \,.\, D_m s \,.\, D_n s. \quad (980)$$

171. *Corollary.* If l, m, n are the rectangular coördinates x, y, z, we have by (725),

$$ds^2 = dx^2 + dy^2 + dz^2 \quad (981)$$

$$a' = a'' = a''' = \tfrac{1}{2}\pi, \quad M = 1; \quad (982)$$

and (980) gives

$$D^3_{x.y.z} V = 1, \quad (983)$$

$$V = \int_x \int_y \int_z 1 = \int_x \int_y z = \int_x \int_z y = \int_y \int_z x. \quad (984)$$

172. *Corollary.* If l, m, n are the polar coördinates of § 73 of B. I., the equations (31, 32, 33) of vol. 1 give, by putting

$$\left.\begin{aligned} u &= r\sin.\varphi \\ y &= u\cos.\theta \\ z &= u\sin.\theta, \end{aligned}\right\} \quad (985)$$

$$dy^2 + dz^2 = du^2 + u^2\,d\theta^2 \quad (986)$$

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= dx^2 + du^2 + u^2\,d\theta^2 \\ &= dr^2 + r^2\,d\theta^2 + r^2\sin.^2\varphi\,d\theta^2, \end{aligned} \quad (987)$$

$$D^3_{r.\varphi.\theta} V = r^2 \sin.\varphi, \quad (988)$$

$$V = \int^2_{r.\varphi} r^2\theta\sin.\varphi = -\int^2_{r.\theta} r^2\cos.\varphi = \tfrac{1}{3}\int^2_{\theta.\varphi} r^3\sin.\varphi. \quad (989)$$

173. *Corollary.* If the coördinates are x, u, θ of the preceding corollary, (987) gives

$$D^3_{x.u.\theta} V = u \quad (990)$$

$$V = \int^2_{x.u} u\theta = \tfrac{1}{2}\int^2_{x.\theta} u^2 = \int^2_{u.\theta} u\,x. \quad (991)$$

Volume of sphere, ellipsoid.

174. *Corollary.* If the given solid is one of revolution about the axis z, of which a segment is required formed by two planes perpendicular to the axis of revolution, z may be substituted for x in (991), and the integrals relative to θ taken from 0 to 2π. Hence

$$V = 2\pi \int^2_{z\,.\,u} .\, u = \pi \int_z .\, u^2 = 2\pi \int_u .\, z\,u. \qquad (992)$$

175. Examples.

1. To find the volume of the segment of a sphere.

Solution. If R is the radius of the sphere, and if the axis of z is perpendicular to the bases of the segment, (992) gives

$$\begin{aligned} V &= \pi \int_z .\, (R^2 - z^2) \\ &= \pi R^2 (z_1 - z_0) - \tfrac{1}{3}\pi (z_1^3 - z_0^3). \qquad (993) \end{aligned}$$

Corollary. The solidity of the sphere is $\frac{4}{3}\pi R^3$. (994)

2. Given the volume V of a solid included within any surfaces whatever, the combination of which, considered as one surface which in general is discontinuous, is represented by the equation

$$F.\,(x\,.\,y\,.\,z) = 0, \qquad (995)$$

to find the volume V' of a solid included within the system of surfaces

$$F.\left(\frac{x}{a},\ \frac{y}{b},\ \frac{z}{c}\right) = 0. \qquad (996)$$

Ans. $V' = a\,b\,c\ V.$ (997)

3. To find the volume of the segment of an ellipsoid included between two planes drawn perpendicular to either of the axes of the ellipsoid.

Ans. If A, B, C are the axes of the ellipsoid, if the planes are drawn perpendicular to the axis of C, and if V is the solidity of the segment of a sphere whose radius is unity, the segment being included between two planes drawn at the distances $\frac{z_0}{C}$ and $\frac{z_1}{C}$ from the centre, the required volume is

$$V' = A\,B\,C\,V. \tag{998}$$

4. To find the volume of the segment of an hyperboloid included between two planes drawn perpendicular to that axis, for which the sections made by the planes are elliptical.

Ans. If C is the axis perpendicular to the planes, and if A and B are the other two axes, the required volume is

$$V = \tfrac{1}{3}\pi\frac{A\,B}{C^3}(z_1^3 - z_0^3) \pm \pi\,A\,B\,(z_1 - z_0), \tag{999}$$

in which the upper sign corresponds to the hyperboloid of one branch, and the lower sign to the hyperboloid of two branches.

5. To find the volume of the segment of the paraboloid, included between two planes drawn perpendicular to the axis of z, the equation of the paraboloid being

$$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 = C\,z. \tag{1 a}$$

$$\textit{Ans.}\quad \tfrac{1}{2}\pi\,A\,B\,C\,(z_1^2 - z_0^2). \tag{2 a}$$

6. To find the volume of the segment of a solid of revolution included between two planes, drawn perpendicular to the axis of revolution, when the revolving arc is that of a cycloid about the axis of x in (130) of vol. 1.

Ans. $V = \tfrac{1}{4}R^2\pi(\sin. 2\,\theta_1 - \sin. 2\,\theta_0) - 2\,R^2\pi(\sin.\,\theta_1 - \sin.\,\theta_0)$
$+ 3\,R^2\pi(\theta_1 - \theta_0).$ (3 a)

7. To find the volume of the segment of the solid of revolution of § 174, when

$$u = B \operatorname{Cos.} \frac{z}{A}. \tag{4 a}$$

Ans. $V = \frac{1}{4} A B^2 \pi \left(\operatorname{Sin.} \frac{2 z_1}{A} - \operatorname{Sin.} \frac{2 z_0}{A} \right) + \frac{1}{2} B^2 \pi (z_1 - z_0).$ (5 a)

176. *Problem. To find the maximum or minimum volume which can be included by a surface drawn under given conditions.*

Solution. Since thé general expression for the volume is reduced to the form of a double integral, this problem is precisely similar in its solution to that of § 165.

177. Examples.

1. *To find the maximum or minimum volume, which can be included within a surface of a given area.*

Solution. Since the double integral (984) is to be a maximum, while that of (811) is to be constant. We have, by § 166, if A is a constant multiplier,

$$1 + A \left(\frac{1}{\varrho} + \frac{1}{\varrho'} \right) = 0 ; \tag{6 a}$$

or

$$\frac{1}{\varrho} + \frac{1}{\varrho'} = - \frac{1}{A}; \tag{7 a}$$

that is, *the surface is one for which the sum of the reciprocals of the greatest and least radii of curvature at each*

point is constant. The general equation of this surface has never been obtained, but *the sphere and the cylinder* are evidently cases of it.

2. To find the solids of revolution which are solutions of the preceding problem.

Solution. Let the axis of z be that of revolution, and by putting

$$u = \sqrt{(x^2 + y^2)}, \tag{8 a}$$

(7 a) becomes, by means of (952),

$$\frac{D_u z + D_u z^3 + u D_u^2 z}{u(1 + D_u z^2)^{\frac{3}{2}}} = -\frac{1}{A} = \frac{D_u z}{u\sqrt{(1+D_u z^2)}} + \frac{D_u^2 z}{(1+D_u z^2)^{\frac{3}{2}}}. \tag{9 a}$$

Let v be taken so that

$$v = \frac{u D_u z}{\sqrt{(1 + D_u z^2)}}, \tag{10 a}$$

whence

$$\log. v = \log. u + \log. D_u z - \tfrac{1}{2} \log. (1 + D_u z^2); \tag{11 a}$$

the differential of which is

$$\begin{aligned} \frac{D_u v}{v} &= \frac{1}{u} + \frac{D_u^2 z}{D_u z} - \frac{D_u z\, D_u^2 z}{1+D_u z^2} \\ &= \frac{1}{u} + \frac{D_u^2 z}{D_u^2 (1+D_u z^2)}; \end{aligned} \tag{12 a}$$

which, multiplied by (10 a), gives by (9 a),

$$D_u v = \frac{v}{u} + \frac{u D_u^2 z}{(1+D_u z^2)^{\frac{3}{2}}} = \frac{D_u z}{\sqrt{(1+D_u z^2)}} + \frac{u D_u^2 z}{(1+D_u z^2)^{\frac{3}{2}}} = -\frac{u}{A}. \tag{13 a}$$

The integral of this equation is

$$v = -\frac{u^2}{2A} + B, \tag{14 a}$$

in which B is an arbitrary constant. But if τ is taken so that

$$D_u z = \cot.\ \tau, \tag{15 a}$$

(10 a and 14 a) give

$$v = u \cos.\ \tau, \tag{16 a}$$

$$\cos.\ \tau = -\frac{u}{2A} + \frac{B}{u}, \tag{17 a}$$

$$\sqrt{\left(\frac{2B}{A} + \cos.^2 \tau\right)} = \frac{u}{2A} + \frac{B}{u}, \tag{18 a}$$

$$u = A \cos.\tau - \sqrt{(2AB + A^2\cos.^2\tau)}, \tag{19 a}$$

$$D_\tau u = -A \sin.\tau + \frac{A^2 \sin.\ \tau \cos.\ \tau}{\sqrt{(2AB + A^2\cos.^2\tau)}}, \tag{20 a}$$

$$D_\tau z = -A \cos.\tau + \frac{A^2 \cos.^2 \tau}{\sqrt{(2AB + A^2\cos.^2\tau)}}. \tag{21 a}$$

If e is taken so that

$$e = \frac{A}{\sqrt{(A^2 + 2AB)}}, \tag{22 a}$$

(21 a) gives, by the notation of elliptic integrals,

$$\sqrt{(2AB + A^2 \cos.^2 \tau)} = \sqrt{(A^2 + 2AB)}\ .\ \Delta\tau. \tag{23 a}$$

$$D_\tau z = -A \cos.\ \tau + \frac{A^2 - 2AB}{\sqrt{(A^2 + 2AB)}.\Delta\tau} + \frac{2AB}{\sqrt{(A^2+2AB)}}\Delta\tau, \tag{24 a}$$

$$z = -A \sin.\ \tau + (A - 2B)\, e F\tau + 2\, e B E\tau\ ; \tag{25 a}$$

and z may be found in in terms of u, by substituting (19 a) in (25 a).

The preceding solution applies strictly to that case only in which A and B have the same sign; for, when they have opposite signs, e becomes greater than unity, and when B is also greater than A, e is imaginary; but these cases are solved without difficulty.

Greatest solid under given conditions.

3. **To find the greatest solid of all those for which**

$$\int_x \int_y \sec.^2 \gamma, \tag{26 a}$$

has a given value, γ being the inclination of the tangent plane of the bounding surface to the given plane of $x\,y$.

Solution. If A is the constant multiplier of (26 a), the equation of the maximum is

$$1 - A^{-1} D_x^2 z - A^{-1} D_y^2 z = 0, \tag{27 a}$$

which is easily derived from the equation

$$\sec.^2 \gamma = 1 + D_x z^2 + D_y z^2. \tag{28 a}$$

Let v be taken so that

$$z = \tfrac{1}{4} A\,(x + y)^2 + v, \tag{29 a}$$

which gives

$$D_x^2 z = \tfrac{1}{2} A + D_x^2 v, \tag{30 a}$$

$$D_y^2 z = \tfrac{1}{2} A + D_y^2 v; \tag{31 a}$$

which, substituted in (27 a), give

$$D_x^2 v + D_y^2 v = 0. \tag{32 a}$$

Let now

$$m = x + y\sqrt{-1}, \quad n = x - y\sqrt{-1}; \tag{33 a}$$

and we have

$$\left.\begin{aligned} D_x v &= D_m v + D_n v, \\ D_y v &= (D_m v - D_n v)\sqrt{1}, \\ D_x^2 v &= D_m^2 v + 2\,D_{m.n}^2 v + D_n^2 v, \\ D_y^2 v &= -D_m^2 v + 2\,D_{m.n}^2 v - D_n^2 v; \end{aligned}\right\} \tag{34 a}$$

which, substituted in (32 a), give

$$D_{m.n}^2 v = 0 \;= D_m \,.\, D_n\, v. \tag{35 a}$$

Hence $D_n\, v$ is a function, whose differential coefficient taken relatively to m is constant, and may, therefore, be any function whatever of n, represented by N; that is,

$$D_n\, v = N. \tag{36 a}$$

Hence $$v = \int_n N = f.n + F.m, \tag{37 a}$$

in which f and F are any *arbitrary functions*; $f.n$ is the function whose differential coefficient is N, and $F.m$ is the arbitrary quantity which is constant relatively to n; that is, which does not vary with n, but may be any function whatever of the other variable m, and which is added to complete the integral. By the substitution of (33 a), (37 a) gives

$$v = f.(x + y\sqrt{-1}) + F(x - y\sqrt{-1}). \tag{38 a}$$

If we put $$F = f' + \sqrt{-1}\,.F' \tag{39 a}$$

$$F = f' - \sqrt{-1}\,.F', \tag{40 a}$$

in which f' and F' are real functions, the value of v becomes

(41 a)

$$v = f'.(x + \sqrt{-1}) + f'.(x - \sqrt{-1}) + \sqrt{-1}\,[F'(x + \sqrt{-1}) - F'(x - \sqrt{-1})],$$

from which the imaginary quantities will wholly disappear.

CHAPTER X.

INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS.

178. A differential equation is said to be of the same *order* with that of the highest differential coefficient which it involves.

The *degree* of a differential equation is determined in the same way as that of an ordinary equation, except that the independent variables are neglected, and each differential coefficient is counted as a variable.

Thus the equation

$$A\,D_x^n\,v + B\,D_x^{n-1}\,v + \&c. + A'D_y^n\,v + B'\,D_y^{n-1}\,v + \&c.$$
$$+ E\,D_x^{n-m}\,.\,D_y^m\,v + \&c. + e\,v + n = 0. \qquad (42\,a)$$

is of the n order; but it is only of the first degree, or *linear*, if the coefficients A, B, &c. involve the independent variables x, y, &c., but do not involve v, &c.

179. Any equation, which is of a less order than a given differential equation, and satisfies it by the aid of differentiation without the assistance of any other equation, is said to be an integral of the given equation. The integral is said to be *complete* when it contains the greatest possible number of arbitrary quantities.

180. *Problem. To integrate several given equations, between the variables x, y, z, &c., and their differential coefficients taken with respect to the independent varia-*

ble t, when the given equations are linear, and contain no term independent of x, y, z, &c., and when all the coefficients are constant, and the number of equations the same with that of x, y, z, &c.

Solution. If the following expressions are assumed for the variables

$$x = A\,e^{st}, \quad y = B\,e^{st} \;\; \&c., \tag{43 a}$$

in which s, A, B, &c., are constant, their differentials give

$$\left.\begin{array}{ll} D_t\,x = A\,s\,e^{st}, & D_t\,y = B\,s\,e^{st}, \;\&c. \\ D_t^2\,x = A\,s^2\,e^{st}, & D_t^2 y = Bs^2\,e^{st}, \;\&c. \\ \&c. & \&c. \end{array}\right\} \tag{44 a}$$

If these values are substituted in the given equations, these equations will evidently become divisible by e^{st}; and the division by this factor will free the equations wholly from variables, and reduce them to equations between s, A, B, &c., in which A and B will have a linear form. If all of the constants A, B, &c. but one, as A, are eliminated, the result will be a single equation involving A and s, in which A, however, will be a factor of the whole equation; so that the division of this equation by A, will lead to a final equation, involving no other unknown quantity but s, and which will serve to determine s. Let the equation for determining s be denoted by

$$S = 0, \tag{45 a}$$

and each root of it will give corresponding values of A, B, &c., or rather of their ratios, and thence values of x, y, &c., which will be integrals of the given equations.

181. *Corollary.* The number of integrals found by the preceding process, will be the same as that of the different roots of the equation (45 a); but all these integrals can be

united into one expression. For it is evident that, if x_s, y_s, &c. denote any one of these systems of integrals,

$$x = L\,x_s + L'\,x_{s'} + \&c., \quad y = L\,y_s + L'\,y_{s'} + \&c. \qquad (46\,a)$$

will also be a system of integrals, in which L, L', &c. will be arbitrary; for the linear form of the given equations will cause the multipliers of L, L', &c. to become th esame functions of x_s, y_s, &c., which the whole equations are of x, y, &c.; and therefore x_s, y_s, &c. will satisfy the equations in the same way as they do when they are by themselves; that is, the aggregate of the terms dependent upon them will be zero.

182. *Corollary.* If the first member of the equation (45 a) is reduced to the form

$$s^n + a\,s^{n-1} + \&c., \qquad (47\,a)$$

the expressions

$$x = \mathcal{E}\frac{A\,e^{st}}{((S))}, \quad y = \mathcal{E}\frac{B\,e^{st}}{((S))}\ \&c. \qquad (48\,a)$$

will, by the notation of the residual calculus, include all the terms of (46 a), provided that the residuation is performed relatively to s, and that A, B, C, &c. assume a new system of values for each root of S. The forms (48 a) might, indeed, be directly applied to the integration of the equation, by performing the differentiation under the sign of residuation.

183. *Corollary.* It may be remarked, that A, B, &c. are integral polynomials in terms of s; and if, indeed, they were not so, the multiplication of each of them by their common denominator would reduce them to such polynomials. Neither of them is a polynomial of a higher degree than the $(n-1)$st; for if either of them were of a higher degree, the division by S would reduce such a term to the form

$$Q\,S + R, \qquad (49\,a)$$

in which R, the remainder of the division, must be of a less degree than S; and (49 a) is by (45 a) reduced to R.

184. *Corollary.* If the values of A, B, &c., in (48 a), change their values for different roots of S, only so far as they should to conform to the sign of residuation, the values (48 a) will involve only one arbitrary constant, which is a factor of each of the quantities A, B, &c. But if this arbitrary constant is reduced to unity; and if A, B, &c. are then multiplied by the polynomial

$$\alpha s^{n-1} + \beta s^{n-2} + \&c., \tag{50 a}$$

in which α, β, &c. are arbitrary constants, the requisite number of arbitrary constants is again introduced into (48 a). The values of (48 a) may, by the process of § 183, be reduced to the form

$$x = \mathcal{E}\frac{(L_x \alpha + M_x \beta + \&c.)e^{st}}{((S))},\; y = \mathcal{E}\frac{(L_y \alpha + M_y \beta + \&c.)e^{st}}{((S))},\; \&c. \tag{51 a}$$

in which L_x, L_y, M_x, M_y, &c. are integral polynomial functions of s, neither of which exceeds the $(n-1)$st degree.

185. *Corollary.* The differentials of (51 a) become, by the same method of reduction,

$$\left.\begin{aligned} D_t\, x &= \mathcal{E}\frac{(L_x \alpha + M_x \beta + \&c.)\, s\, e^{st}}{((S))} \\ &= \mathcal{E}\frac{(L'_x \alpha + M'_x \beta + \&c.)\, e^{st}}{((S))} \\ D_t^2\, x &= \mathcal{E}\frac{(L''_x \alpha + M''_x \beta + \&c.)\, e^{st}}{((S))} \\ D_t\, y &= \mathcal{E}\frac{(L'_y \alpha + M'_y \beta + \&c.)\, e^{st}}{((S))} \\ \&c. \end{aligned}\right\} \tag{52 a}$$

186. *Corollary.* If x_0, y_0, &c.; x'_0, y'_0, &c.; x''_0, &c., represent the values of x, y, &c.; $D_t\,x$, $D_t\,y$, &c.; $D_t^2 x$, &c. when t vanishes; and if

$$\left.\begin{array}{l} H_x = \mathcal{E}\dfrac{L_x}{((S))},\quad H'_x = \mathcal{E}\dfrac{L'_x}{((S))},\ \&\text{c.},\ I_x = \mathcal{E}\dfrac{M_x}{((S))} \\ H_y = \mathcal{E}\dfrac{L_y}{((S))},\quad I_y = \mathcal{E}\dfrac{M_y}{((S))},\ \&\text{c.}, \end{array}\right\} \quad (53\text{ a})$$

equations (51 a and 52 a) give

$$\left.\begin{array}{ll} x_0 = H_x\,\alpha + I_x\,\beta + \&\text{c.}, & y_0 = H_y\,\alpha + I_y\,\beta + \&\text{c.}; \\ x'_0 = H'_x\,\alpha + I'_x\,\beta + \&\text{c.}, & \&\text{c.} \end{array}\right\} \quad (54\text{ a})$$

If the number of the equations (54 a) is taken equal to that of the constant α, β, &c., the values of α, β, &c., may, by the usual process of elimination, be found in terms of x_0, y_0, &c. The expressions of α, β, &c. in terms of x_0, y_0, &c. will clearly be linear functions of α, β, &c.; so that if these values are substituted in (51 a and 52 a), the expressions of x, y, &c. will contain x_0, y_0, &c., in the same linear form in which they now contain α, β, &c. The values of α, β, &c., in (51 a and 52 a), might, then, have been assumed at once as identical with x_0, y_0, &c., and the corresponding values of x, y, &c. would be

$$\left.\begin{array}{l} x = \mathcal{E}\dfrac{L_x\,x_0 + L'_x\,x'_0 + \&\text{c.} + M_x\,y_0 + \&\text{c.}}{((S))}e^{st}, \\ y = \mathcal{E}\dfrac{L_y\,x_0 + L'_y\,x'_0 + \&\text{c.} + M_y\,y_0 + \&\text{c.}}{((S))}e^{st}, \\ \quad\&\text{c.}; \end{array}\right\} \quad (55\text{ a})$$

in which, it may be observed, that the values of L_x, M_x, &c. are entirely distinct from those in (51 a and 52 a).

187. *Lemma.* If F' denotes the value which $x f . x$ acquires when x becomes infinite, we have

$$F' = \mathcal{E}((f . x)), \tag{56 a}$$

whenever $f . x$ denotes a rational fraction, of which the degree of the numerator is less than that of the denominator.

Proof. It follows from (216 and 219), that, in the present case,

$$f . x = \mathcal{E}\frac{((f . z))}{x - z}, \tag{57 a}$$

the product of which by x is

$$x f . x = \mathcal{E}\frac{x\,((f . z))}{x - z}. \tag{58 a}$$

But when x is infinite, (58 a) becomes

$$F' = \mathcal{E}((f . z)) = \mathcal{E}((f . x)). \tag{59 a}$$

188. *Corollary.* When the excess of the degree of the denominator of $f . x$ above the numerator is greater than unity, (59 a) becomes

$$0 = \mathcal{E}((f . x)). \tag{60 a}$$

189. *Corollary.* When the excess of the degree of the denominator of $f . x$ above the numerator is exactly unity, and when $f . x$ is of the value (217), (59 a) becomes

$$\frac{a}{a'} = \mathcal{E}((f . x)). \tag{61 a}$$

190. *Corollary.* Since, when t becomes zero, the values of x, y, &c. (55 a) are reduced to x_0, y_0, &c.; the polyno-

mials L'_x, M_x, L_y, L'_y, &c. must be of a less degree than the $(n-1)$th; while L_x, M_y, &c. must be of the form

$$s^{n-1} + b\, s^{n-2} + \&c. \qquad (62\text{ a})$$

The form of $s\, L_x$ is, therefore,

$$s^n + b\, s^{n-1} + \&c.\,; \qquad (63\text{ a})$$

so that, by (47 a), $\qquad s\, L_x - S \qquad$ (64 a)

is of a less degree than S. We have, then, by denoting (64 a) by $L_{x'}$,

$$D_t\, x = \mathcal{E}\, \frac{L_{x'}\, x_0 + s\, L'_x\, x'_0 + \&c. + s\, M_x\, y_0}{((S))}\, e^{st}. \qquad (65\text{ a})$$

But when t vanishes, $D_t\, x$ is reduced to x'_0, and therefore $s\, L'_x$ must be of the form (62 a), while $L_{x'}$, $s\, M_x$, &c. must be of a smaller degree. We have then, again, by the differentiation of (65 a),

$$D^2 x = \mathcal{E}\, \frac{s\, L_{x'}\, x_0 + (s^2\, L'_x - S) x'_0 + \&c. + s^2\, M_x\, y_0}{((S))}\, e^{st}, \qquad (66\text{ a})$$

and a similar train of argument may be continued to the higher differential coefficients.

191. *Corollary*. If, in the given equations, there are substituted for x, $D_t\, x$, $D_t^2\, x$, &c., the quantities contained under the sign of residuation in (55 a, 65 a, 66 a, &c.), those equations must be satisfied. The reverse process, therefore, of substituting for $D_t\, x$, $D_t\, y$, &c., not $s\, x$, $s\, y$, &c., but $s\, x - x_0\, S \,.\, \frac{e^{st}}{S}$, $s\, y - y_0\, S\, \frac{e^{st}}{S}$, &c., and for $D_t^2\, x$, &c., $s^2\, x - (x'_0 + s\, x_0)\, S\, \frac{e^{st}}{S}$, &c., must give again the parts of x, y, &c. in (55 a), which are under the sign of residuation.

192. *Corollary*. If $\wp$ be taken to denote the expression for S when D_t is substituted for s, and if $\mathfrak{L}$, $\mathfrak{M}$, &c., denote the expressions which L, M, &c. assume by the same substitution, and if Θ be taken so that

$$\Theta = \mathcal{E}\frac{e^{st}}{((S))}, \tag{67 a}$$

we shall have

$$\mathcal{E}\frac{L\,e^{st}}{((S))} = \mathfrak{L}\,\Theta, \tag{68 a}$$

$$\wp\,\Theta = \mathcal{E}S\frac{e^{st}}{((S))} = 0\,; \tag{69 a}$$

and the values of x, y, &c. (55 a) will become

$$\left.\begin{array}{l} x = (\mathfrak{L}_x\, x_0 + \mathfrak{L}'_x\, x'_0 + \text{\&c.} + \mathfrak{M}_x\, y_0 + \text{\&c.})\,\Theta, \\ y = (\mathfrak{L}_y\, x_0 + \mathfrak{L}'_y\, x'_0 + \text{\&c.} + \mathfrak{M}_y\, y_0 + \text{\&c.})\,\Theta, \\ \quad\text{\&c.,} \end{array}\right\} \tag{70 a}$$

in which $\mathfrak{L}_x$ &c. are not proper factors, but express functional operations to be performed.

The value of $\wp$ would be obtained by eliminating x, y, &c. directly from the given equation, in which process D_t, D_t^2, &c. are to be treated as though they were factors. The values of x, y, &c. (70 a), will then be obtained by the same process of elimination, from the equations, which are obtained from the given equations, by substituting

$$\left.\begin{array}{ll} D_t\, x - x_0\,\wp\,\Theta & \text{for } D_t\, x, \\ D_t\, y - y_0\,\wp\,\Theta & \text{for } D_t\, y, \text{ \&c.} \\ D_t^2\, x - (x'_0 + x_0\, D_t)\,\wp\,\Theta, & \\ \quad\text{\&c.} & \end{array}\right\} \tag{71 a}$$

This is *Cauchy's method of integration*, and the function Θ is called the *principal function*.

193. *Corollary.* When the equation (45 a) has several equal roots, the corresponding systems of values in (46 a) would seem to coalesce into one. This loss of terms, and therefore of arbitrary constants, is, however, unnecessary; for if the roots s, s', s'', &c., instead of being equal, differed infinitely little from each other, so that

$$s' = s + h, \quad s'' = s' + h' = s + h + h', \text{ \&c.,} \qquad (72\text{ a})$$

in which h, h', &c. are infinitely small, we shall have, by (416) of vol. 1, upon putting

$$B = A' h, \quad B' = A'' h', \quad C = B' h, \text{ \&c.,} \qquad (73\text{ a})$$

$$\left.\begin{aligned} A' e^{s't} &= A' e^{st+ht} = A' e^{st} e^{ht} = A' e^{st} + A' h\, t\, e^{st} = A' e^{st} + B\, t\, e^{st} \\ A'' e^{s''t} &= A'' e^{s't} + B' t\, e^{s't} = A'' e^{st} + (A'' h + B') t\, e^{st} + C\, t^2\, e^{st} \end{aligned}\right\} \qquad (74\text{ a})$$

&c.

The new terms, multiplied by t, t^2, &c., which are thus introduced, are just sufficient to replace those which are lost by addition. These very terms are also introduced by the process of residuation, for this process requires, by (182), one or more differentiations, whenever the roots are equal, and each differentiation will have to be applied to e^{st} in (55 a). But by (481) of vol. 1,

$$D_s^m\, e^{st} = t^m\, e^{st}, \qquad (75\text{ a})$$

whence the differentiation will, evidently, introduce the required terms.

194. *Problem.* *To integrate several linear differential equations between the variables x, y, &c., and their differential coefficients taken relatively to the independent variable t, when all the coefficients are constant, the terms which are independent of x, y, &c. are given functions of t, and the number of the equations is the same with that of the variables.*

Linear differential equations with constant coefficients.

Solution. When the functions of t are reduced to zero, this problem coincides with the preceding one; and if ξ, η, &c. denote the corresponding values (70 a) of x, y, &c. obtained by the preceding process; while X, Y, &c. are particular values of x, y, &c., which satisfy the present problem, the values

$$x = \xi + X, \quad y = \eta + Y, \text{ \&c.} \tag{76 a}$$

are complete values of x, y, &c. for ξ, η, &c., involve the required number of arbitrary constants.

The problem is reduced, then, to obtaining these particular values of x, y, &c. For this purpose, let the subsidiary quantity τ be introduced, and let

$$\Theta_\tau = e^{-s\tau}\,\Theta = \mathfrak{L}\frac{s^{(t-\tau)}}{((S))}, \tag{77 a}$$

so that Θ_τ is the value which Θ assumes when t is changed into $t-\tau$. If, then, $\mathfrak{X}$, $\mathfrak{Y}$, &c. are the values, which ξ, η, &c. assume, when Θ is changed to Θ_τ, and when for x_0, x'_0, y_0, &c. are substituted $\mathfrak{T}_x$, $\mathfrak{T}'_x$, $\mathfrak{T}_y$, &c., which are functions of τ, and if the integrations in the following formulas are performed relatively to τ, we may put

$$X = \int_0^t \mathfrak{X}, \quad Y = \int_0^t \mathfrak{Y}, \text{ \&c.} \tag{78 a}$$

The differentiation of (78 a) relatively to t, involves not only the differentiation of $\mathfrak{X}$, $\mathfrak{Y}$, &c., under the sign of integration, but also the changes arising from the change in the limits of integration. If then $\mathfrak{X}_t$, $\mathfrak{Y}_t$, &c., are the values which $\mathfrak{X}$, $\mathfrak{Y}$, &c., assume when τ is changed to t, the differentiation of (78 a) gives

$$\left.\begin{aligned} D_t X &= \mathfrak{X}_t + \int_0^t D_t\,\mathfrak{X}, \\ D_t Y &= \mathfrak{Y}_t + \int_0^t D_t\,\mathfrak{Y}, \text{ \&c.} \end{aligned}\right\} \tag{79 a}$$

If, again, we put

$$\mathfrak{x}' = D_t \mathfrak{x}, \quad \mathfrak{y}' = D_t \mathfrak{y}, \tag{80 a}$$

another differentiation of (79 a) gives

$$D_t^2 X = D_t \mathfrak{x}_t + \mathfrak{x}'_t + \int_0^t D_t^2 \mathfrak{x}, \text{ \&c.} \tag{81 a}$$

By the substitution of X, Y, &c. for x, y, &c., in the given equations, the terms under the sign of integration must disappear, for the terms under this sign in the values of X, $D_t Y$, &c. differ from the values of ξ, η, &c. in nothing but the common factor $e^{-s\tau}$, and the writing of the particular forms $\mathfrak{T}_x$, $\mathfrak{T}'_x$, &c. for the arbitrary constants x_0, x'_0, &c.

The substitution of t for τ reduces $e^{s(t-\tau)}$ to unity, and if T_x, T'_x, are the same functions of t, which $\mathfrak{T}_x$ and $\mathfrak{T}'_x$ are of τ, we have, by § 190,

$$\left.\begin{aligned}
\mathfrak{x}_t &= \mathcal{E}\frac{L_x\, T_x + L'_x\, T'_x + \text{\&c.} + M_x\, T_y + \text{\&c.}}{((S))} = T_x \\
\mathfrak{x}'_t &= \mathcal{E}\frac{L_{x'}\, T_x + s\, L'_x\, T'_x + \text{\&c.} + s\, M_x\, T_y + \text{\&c.}}{((S))} = T'_x, \text{ \&c.} \\
\mathfrak{y}_t &= T_y, \quad \text{\&c.}
\end{aligned}\right\} \tag{82 a}$$

Hence, by the omission of the parts of (78 a, 79 a and 81 a), which are under the signs of integration, they become

$$\left.\begin{aligned}
&X = 0, \quad D_t X = T_x, \quad D_t^2 X = T'_x + D_t . T_x, \text{ \&c.} \\
&Y = 0, \quad D_t Y = T_y, \qquad \text{\&c.};
\end{aligned}\right\} \tag{83 a}$$

and the substitution of these values in the given equations, reduces them to linear differential equations in which T_x, T'_x, T_y are the variables, and the order of the equations is less by one than that of the given equations. Thus the number of these variables being greater than that of the equations,

enables us to take certain of them at pleasure. Thus of th quantities T_x, T'_x, &c., all but one may be supposed to b zero; of T_y, T'_y, all but one may be zero; and in the sam way with the others.

The selection of the quantities T_x, &c., which are to remain of a finite value, is immediately fixed, by the consideration that the resulting equation should be of as low an order as possible. It is generally possible to select those quantities which correspond, respectively, to the highest order of differential coefficients of x, y, &c.; and with this selection the resulting equations are wholly free from differentials, and are solved by simple elimination. In any case, however, it seems possible to make a selection which will avoid the necessity of integration.

195. *Corollary.* When s is nothing, the values of X, Y must vanish, as well as all their differential coefficients of an order inferior to those which correspond to the quantities in the series T_x, T_y, &c., which are retained as finite. Hence the corresponding values of x, y, $D_t x$, &c. will be reduced to x_0, y_0, x'_0, &c.

196. Examples.

1. To integrate the differential equation

$$D_t^n x + a\, D_t^{n-1} x + \&c. = U, \qquad (84\text{ a})$$

in which U is a function of t.

Solution. In this case, the value of φ becomes at once

$$\varphi = D_t^n + a\, D_t^{n-1} + \&c. \quad = f.D_t\,; \qquad (85\text{ a})$$

Hence $$S = s^n + a\,s^{n-1} + \&c. \quad = f.s, \tag{86 a}$$

by taking $f.$ to denote the integral function, which constitutes the second members of (85 a and 86 a). We have also

$$\Theta = \mathcal{E}\frac{e^{st}}{S} = \mathcal{E}\frac{e^{st}}{f.s}, \tag{87 a}$$

and the equation for determining ξ is, by (71 a),

$$(D_t^n + a\,D_t^{n-1} + \&c.)\xi$$
$$-[x_0(D_t^{n-1} + a\,D_t^{n-2} + \&c.) + x_0'\,(D_t^{n-2} + a\,D_t^{n-3} + \&c.) + \&c.]\,\digamma\,\Theta = 0$$

or (88 a)

$$\digamma\,\xi - [x_0(D_t^{n-1} + aD_t^{n-2} + \&c.) + x_0'(D_t^{n-2} + aD_t^{n-3} + \&c.) + \&c.]\,\digamma\,\Theta = 0;$$

whence (89 a)

$$\xi = [x_0(D_t^{n-1} + a\,D_t^{n-2} + \&c.) + x_0'(D_t^{n-2} + a\,D_t^{n-3} + \&c.) + \&c.]\Theta.$$

If, in the development of the expression

$$\frac{D_t^n - x_0^n}{D_t - x_0}, \tag{90 a}$$

the exponents n, $n-1$, &c. of x_0, are regarded as expressing the number of accents, and if the term which does not contain x_0 is multiplied by x_0 the value (89 a) may be expressed in a more simple form; for we shall have

$$\xi = \left(\frac{D_t^n - x_0^n}{D_t - x_0} + \frac{a\,(D_t^{n-1} - x_0^{n-1})}{D_t - x_0} + \&c.\right)\Theta$$
$$= \frac{f.D_t - f.x_0}{D_t - x_0}\,\Theta. \tag{91 a}$$

To obtain the value of X, let $\mathfrak{U}$ be the value which U assumes when t is changed to τ, and by omitting the accent of T_x as unnecessary, we have by § 194, if we suppose all the quantities in the series T_x, &c. to vanish but the $(n-1)$st,

$$T_x = U, \quad \mathfrak{T}_x = \mathfrak{U}; \tag{92 a}$$

whence, by (78 a, 76 a and 91 a),

$$X = \int_0^t \mathfrak{U}\, \Theta_\tau \tag{93 a}$$

$$x = \frac{f.\,D_t - f\,x_0}{D_t - x_0}\Theta + \int_0^t \mathfrak{U}\, \Theta_\tau. \tag{94 a}$$

2. To integrate the differential equation

$$D_t^3 x - (a+b)\, D_t^2 x + a\,b\, D_t x = c\,t.$$

Solution. In this case, we have

$$f.D_t = D_t^3 - (a+b)\, D_t^2 + a\,b\, D_t = D_t\,(D_t - a)\,(D_t - b)$$

$$S = s\,(s-a)\,(s-b)$$

$$\frac{f.D_t - f.x_0}{D_t - x_0} = x_0[D_t^2 - (a+b)\,D_t + a\,b] + x_0'\,[D_t - (a+b)] + x_0''$$

$$\xi = \frac{f.D_t - f.x_0}{D_t - x_0}\Theta = \mathcal{E}\,\frac{x_0[s^2-(a+b)s+a\,b]+x_0'\,[s-(a+b)]+x_0''}{((s\,(s-a)\,(s-b)))}e^{st}$$

$$= x_0 - \frac{a+b}{a\,b}\,x_0' + \frac{1}{a\,b}\,x_0''$$

$$-\frac{b\,x_0' + x_0''}{a\,(a-b)}\,e^{at} + \frac{a\,x_0' - x_0''}{b\,(a-b)}\,e^{bt},$$

$$\mathfrak{U} = c\,\tau,$$

$$X = \int_0^t \mathfrak{U}\,\Theta_\tau = \mathcal{E}\,\frac{\int_0^t c\,\tau\, e^{s\,(t-\tau)}}{((s(s-a)(s-b)))} = \mathcal{E}\,\frac{c\,e^{st} - c\,(st+1)}{s^2\,((s(s-a)\,(s-b)))}$$

$$= \frac{c\,t^2}{2\,a\,b} + \frac{c\,e^{at} - c(at+1)}{a^3\,(a-b)} - \frac{c\,e^{bt} - c(bt+1)}{b^3\,(a-b)}$$

and $\quad x = \xi + X.$

3. To integrate the differential equation

$$D_t^3\, x - 3\,a\, D_t^2 x + 3\,a^2\, D_t\, x - a^3\, x = b\,e^{at}.$$

Solution. In this case, we have

$$f.D_t = (D_t - a)^3$$

$$S = (s - a)^3$$

$$\frac{f.D_t - f.x_0}{D_t - x_0} = x_0 (D_t - a)^2 + (x_0' - a\,x_0)(D_t - a) + (x_0'' - 2\,a\,x_0' + a^2 x_0)$$

$$\xi = \mathcal{E}\frac{x_0\,(s-a)^2 + (x_0' - a\,x_0)\,(s-a) + (x_0'' - 2\,a\,x_0' + a^2\,x_0)}{((s-a))^3}e^{st}$$

$$= [x_0 + t\,(x_0' - a\,x_0) + \tfrac{1}{2}\,t^2\,(x_0'' - 2\,a\,x_0' + a^2 x_0)]e^{at}$$

$$\mathfrak{X} = b\,e^{a\tau}$$

$$X = \mathcal{E}\frac{\int_0^t b\,e^{a\tau}\,e^{s(t-\tau)}}{((s-a))^3} = \mathcal{E}\frac{b\,(e^{at} - e^{st})}{(a-s)\,((s-a))^3} = \tfrac{1}{6}\,b\,t^3\,e^{at}.$$

4. To integrate the differential equation

$$D_t^3\,x + a^2\,D_t\,x = b\,\sin.\,m\,t.$$

Solution. In this case, we have

$$f.D_t = (D_t^3 + a^2\,D_t) = D_t\,(D_t^2 + a^2)$$

$$S = s\,(s^2 + a^2)$$

$$\frac{f.D_t - f.x_0}{D_t - x_0} = x_0\,(D_t^2 + a^2) + x_0'\,D_t + x_0''$$

$$\xi = \mathcal{E}\frac{x_0\,(s^2 + a^2) + x_0'\,s + x_0''}{((s\,(s^2 + a^2)))}e^{st}$$

$$= x_0 + \frac{x_0''}{a^2} - \frac{x_0'\,a\sqrt{-1} + x_0''}{2\,a^2}e^{at\sqrt{-1}} + \frac{x_0'\,a\sqrt{-1} - x_0''}{2\,a^2}e^{-at\sqrt{-1}}$$

$$= x_0 + \frac{x_0'}{a}\,\sin.\,a\,t + \frac{x_0''}{a^2}\,(1 - \cos.\,a\,t)$$

$$= x_0 + \frac{x_0'}{a}\,\sin.\,a\,t + \frac{2\,x_0''}{a^2}\,\sin.^2\,\tfrac{1}{2}\,a\,t$$

$$X = \mathcal{E}\frac{\int_0^t b \sin. m\tau . e^{s(t-\tau)}}{((s(s^2+a^2)))} = \mathcal{E}\frac{-s\,b \sin. m\,t - m\,b\,(\cos. m\,t + e^{st})}{(m^2+a^2)((s(s^2+a^2)))}$$

$$= \frac{b}{m\,a^2} + \frac{m\,b}{m^2-a^2}\left(\frac{\cos. m\,t}{m^2} - \frac{\cos. a\,t}{a^2}\right).$$

$$x = \xi + X.$$

Corollary. When m is equal to a, the value of X becomes

$$X = \frac{b}{a^3}(1-\cos. a\,t) - \frac{bt \sin. at}{2\,a^2} = \frac{2b}{a^3}\sin. \tfrac{1}{2}\,a\,t\,(\sin. \tfrac{1}{2}a\,t - \tfrac{1}{2}\,a\,t \cos. \tfrac{1}{2}a\,t).$$

5. To integrate the differential equation

$$D_t^2\,x - (a+b)\,D_t\,x + a\,b\,x = h\,t^2 + k\,e^{m\,t} + l \sin. n\,t.$$

Ans.
$$x = \frac{(x_0' - b\,x_0)e^{at} - (x_0' - a\,x_0)e^{bt}}{a-b} + \frac{2\,h\,(a^2+a\,b+b^2)}{a^3\,b^3}$$

$$+ \frac{2\,h\,(a+b)t}{a^2\,b^2} + \frac{h\,t^2}{a\,b} + \frac{2\,h\,e^{at}}{a^3\,(a-b)} - \frac{2\,h\,e^{bt}}{b^3\,(a-b)}$$

$$+ \frac{k\,e^{mt}}{(m-a)\,(m-b)} - \frac{k\,e^{at}}{(m-a)\,(a-b)} + \frac{k\,e^{bt}}{(m-b)\,(a-b)}$$

$$+ \frac{l(a\,b - n^2) \sin. n\,t + n\,l\,(a+b) \cos. n\,t}{(a^2+n^2)\,(b^2+n^2)}$$

$$+ \frac{l\,n\,e^{at}}{(a^2+n^2)\,(a-b)} - \frac{l\,n\,e^{bt}}{(b^2+n^2)\,(a-b)}.$$

When $m = a$, the terms multiplied by k become

$$\frac{k\,t\,e^{at}}{a-b} - \frac{k\,(e^{at} - e^{bt})}{(a-b)^2};$$

when $m = b$, the terms multiplied by k become

$$-\frac{k\,t\,e^{bt}}{a-b} + \frac{k\,(e^{bt} - e^{at})}{(a-b)^2}.$$

Linear differential equations with constant coefficients.

6. To integrate the differential equation

$$D_t^2 x - 2a\, D_t x + a^2 x = h t^2 + k e^{mt} + l \sin. n t.$$

Ans. $x = x_0 e^{at} + (x_0' - a x_0)\, t e^{at}$

$+ \frac{h}{a^4} [6 + 4at + a^2 t^2 + 2(at - 3) e^{at}]$

$+ \frac{k}{(m-a)^2} [e^{mt} - e^{at} - (m - a)\, t e^{at}]$

$+ \frac{l}{(n^2+a^2)^2}[(a^2-n^2)\sin. nt + (a^2+n^2) n t e^{at} + 2an(\cos. nt - e^{at})]$.

When $m = a$, the terms multiplied by k become

$$\tfrac{1}{2} k t^2 e^{at}.$$

7. To integrate the differential equation

$$D_t^2 x + a^2 x = h t^2 + k e^{mt} + l \sin. n t.$$

Ans. $x = \frac{x_0'}{a} \sin. at + x_0 \cos. at + \frac{h}{a^4} (a^2 t^2 - 4 \sin.^2 \tfrac{1}{2} a t)$

$$+ \frac{k(a e^{mt} - a \cos. at - m \sin. at)}{a(m^2 + a^2)} + \frac{l(n \sin. at - a \sin. nt)}{a(n^2 - a^2)}.$$

When $n = a$, the term multiplied by l becomes

$$\frac{1}{2a^2} (\sin. at - at \cos. at).$$

8. To integrate the differential equation

$$D_t^4 x = x. \tag{95 a}$$

Ans. $x = \tfrac{1}{4}(x_0 + x_0'')(e^t + e^{-t}) + \tfrac{1}{4}(x_0' + x_0''')(e^t - e^{-t})$

$+ \tfrac{1}{2}(x_0 - x_0'') \cos. t + \tfrac{1}{2}(x_0' - x_0''') \sin. t$

$= \tfrac{1}{2}(x_0 + x_0'') \text{Cos.}\, t + \tfrac{1}{2}(x_0' + x_0''') \text{Sin.}\, t$

$+ \tfrac{1}{2}(x_0 - x_0'') \cos. t + \tfrac{1}{2}(x_0' - x_0''') \sin. t.$ (96 a)

Linear differential equations with constant coefficients.

9. To integrate the differential equation

$$D_t^4 x + x = 0. \tag{97 a}$$

Ans. $x = u x_0 + D_t^3 u . x_0' + D_t^2 u . x_0'' + D_t u . x_0'''$, (98 a)

in which $u = \text{Cos.} (\sqrt{\tfrac{1}{2}} . t) \cos. (\sqrt{\tfrac{1}{2}} . t)$. (99 a)

10. To integrate the differential equation

$$D_t^n x = x. \tag{1 b}$$

Ans. $x = u x_0 + D_t^{n-1} u . x_0' + D_t^{n-2} u . x_0'' + \&c.$, (2 b)

in which, when n is an odd number,

$$u = \frac{1}{n}\left[e^t + 2\,\Sigma . e^{t \cos. \frac{2 m \pi}{n}} \cos. \left(t \sin. \frac{2 m \pi}{n}\right)\right], \tag{3 b}$$

where Σ denotes the sum of all the terms which are obtained by substituting for m all the integers from 1 to $\frac{1}{2}(n-1)$ inclusive. But when n is an even number, which is not divisible by 4,

$$u = \frac{2}{n}\left[\text{Cos.}\, t + 2\,\Sigma . \text{Cos.} \left(t \cos. \frac{2 m \pi}{n}\right) . \cos. \left(t \sin. \frac{2 m \pi}{n}\right)\right], \tag{4 b}$$

where Σ denotes the sum of all the terms which are obtained by substituting for m all the integers from 1 to $\frac{1}{2}(\frac{1}{2}n - 1)$ inclusive. When n is divisible by 4,

$$u = \frac{2}{n}\left[\text{Cos.}\, t + \cos.\, t + 2\,\Sigma . \text{Cos.} \left(t \cos. \frac{2 m \pi}{n}\right) . \cos. \left(t \sin. \frac{2 m \pi}{n}\right)\right], \tag{5 b}$$

where Σ denotes the sum of all the terms which are obtained by substituting for m all the integers from 1 to $\frac{1}{2}n - 1$ inclusive.

11. To integrate the differential equation

$$D_t^n x + x = 0. \tag{6 b}$$

Ans. $x = u\,x_0 + D_t^{n-1} u \,.\, x_0' + D_t^{n-2} u \,.\, x_0'' + \&c.,$ (7 b)

in which, when n is an odd number,

$$u = \frac{1}{n}\left[e^{-t} + 2\,\Sigma\,.\,e^{t \cos.\,(2m-1)\frac{\pi}{n}} \cos.\left(t \sin.\,(2m-1)\frac{\pi}{n} \right)\right] \tag{8 b}$$

where Σ is used as in (3 b). When n is an even number, which is not divisible by 4,

$$u = \frac{2}{n}\left[\cos.\,t + 2\,\Sigma.\mathrm{Cos.}\left(t \cos.\,(2m-1)\frac{\pi}{n}\right) \cos.\left(t \sin.\,(2m-1)\frac{\pi}{n}\right)\right] \tag{9 b}$$

where $\Sigma.$ is used as in (4 b). When n is divisible by 4,

$$u = \frac{4}{n}\left[\Sigma.\mathrm{Cos.}\left(t \cos.(2m-1)\frac{\pi}{n}\right) \cos.\left(t \sin.\,(2m-1)\frac{\pi}{n}\right) \right] \tag{10 b}$$

where $\Sigma.$ denotes the sum of all the terms which are obtained by substituting for m all the integers from 1 to $\frac{1}{4}n$ inclusive.

12. To integrate the differential equations

$$\left.\begin{aligned} D_t^2 x + a\,x + b\,y &= X \\ D_t^2 y + a' x + b' y &= Y, \end{aligned}\right\} \tag{11 b}$$

in which X and Y are functions of t.

Ans. In this case we have

$$\digamma = (D_t^2 . + a)\,(D_t^2 + b') - a' b \tag{12 b}$$

$$\mathcal{S} = (s^2 + a)\,(s^2 + b') - a' b \tag{13 b}$$

$$\xi = [(D_t^2 + b')x_0' + (D_t^3 + b' D_t)x_0 - b\,y_0' - b\,y_0\,D_t]\,\Theta \tag{14 b}$$

$$\eta = [(D_t^2 + a)y_0' + (D_t^3 + a\,D_t)y_0 - a' x_0' - a' x_0\,D_t]\,\Theta; \tag{15 b}$$

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and if

$$m = \tfrac{1}{2}(a + b'), \quad n^2 = \tfrac{1}{4}(a - b')^2 + a' b,$$

$$\xi = \frac{(b'-m+n)x'_0 - b\,y'_0}{2\,n\,\sqrt{(m-n)}} \sin.\, t\sqrt{(m-n)} - \frac{(b'-m-n)x'_0 - b\,y'_0}{2\,n\,\sqrt{(m+n)}} \sin.\, t\sqrt{(m+n)}$$

$$+ \frac{(b'-m+n)x_0 - b\,y_0}{2\,n} \cos.\, t\sqrt{(m-n)} - \frac{(b'-m-n)x_0 - b\,y_0}{2\,n} \cos.\, t\sqrt{(m+n)} \quad (16\,b)$$

$$\eta = \frac{(a-m+n)\,y'_0 - a'\,x'_0}{2\,n\,\sqrt{(m-n)}} \sin.\, t\sqrt{(m-n)}$$

$$- \frac{(a-m-n)\,y'_0 - a'\,x'_0}{2\,n\,\sqrt{(m+n)}} \sin.\, t\sqrt{(m+n)}$$

$$+ \frac{(a-m+n)\,y_0 - a'\,x_0}{2\,n} \cos.\, t\sqrt{(m-n)}$$

$$- \frac{(a-m-n)\,y_0 - a'\,x_0}{2\,n} \cos.\, t\sqrt{(m+n)}. \quad (17\,b)$$

If, also, $\mathfrak{X}$, $\mathfrak{Y}$ are the values of X, Y when t is changed to τ, and if the integrals in the following expression are taken relatively to τ,

$$\xi' = \int_0^t . \frac{(b'-m+n)\,\mathfrak{X} - b\,\mathfrak{Y}}{2\,n\,\sqrt{(m-n)}} \sin.\,(t-\tau)\sqrt{(m-n)}$$

$$- \int_0^t . \frac{(b'-m-n)\,\mathfrak{X} - b\,\mathfrak{Y}}{2\,n\,\sqrt{(m+n)}} \sin.\,(t-\tau)\sqrt{(m+n)} \quad (18\,b)$$

$$\eta' = \int_0^t . \frac{(a-m+n)\,\mathfrak{Y} - a'\,\mathfrak{X}}{2\,n\,\sqrt{(m-n)}} \sin.\,(t-\tau)\sqrt{(m-n)}$$

$$- \int_0^t . \frac{(a-m-n)\,\mathfrak{Y} - a'\,\mathfrak{X}}{2\,n\,\sqrt{(m+n)}} \sin.\,(t-\tau)\sqrt{(m+n)}, \quad (19\,b)$$

we have $$x = \xi + \xi', \qquad y = \eta + \eta'. \quad (20\,b)$$

When $(m-n)$ is negative, $\dfrac{\sin.\sqrt{(m-n)}\,t}{\sqrt{(m-n)}}$ and $\cos.\sqrt{(m-n)}\,t$ are to be changed to $\dfrac{\text{Sin}.\sqrt{(n-m)}\,t}{\sqrt{(n-m)}}$ and $\text{Cos}.\sqrt{(n-m)}\,t$. (21 b)

When $m+n$ is negative, $\dfrac{\sin.\sqrt{(m+n)}\,t}{\sqrt{(m+n)}}$ and $\cos.\sqrt{(m+n)}\,t$ are to be changed to $\dfrac{\text{Sin}.\sqrt{-(m+n)}\,t}{\sqrt{-(m+n)}}$ and $\text{Cos}.\sqrt{-(m+n)}\,t$. (22 b)

When m and n are equal, $\dfrac{\sin.\sqrt{(m-n)}\,t}{\sqrt{(m-n)}}$ and $\cos.\sqrt{(m-n)}\,t$ are to be changed to t and unity. (23 b)

When $m+n$ is zero, $\dfrac{\sin.\sqrt{(m+n)}\,t}{\sqrt{(m+n)}}$ and $\cos.\sqrt{(m+n)}\,t$ are to be changed to t and unity. (24 b)

The changes, which correspond to the case when n is zero, are easily made.

197. *Definition. A fluctuating function*, is one which constantly changes its value by a finite quantity for an infinitely small change in the variable, alternately increasing and decreasing without ever being infinite.

This singular function is of great use in the integration of equations which involve several independent variables; there is no name in general use, but the one here adopted was given by Hamilton, and is highly appropriate.

The expression $\sin.\,a\,x$, is an instance of such a function, when a is infinite; and, in this instance, it is noticeable that the mean value of the function is zero.

198. *Theorem. If f_α denotes a function of α which is continuous and finite within the limits a and b, and if N_α is a fluctuating function of which the mean value corresponding to each fluctuation is zero, and if the integrations are performed relatively to α, we have*

$$\int_a^b . N_\alpha \, f_\alpha = 0. \tag{25 b}$$

Proof. Let the interval between the limits of the integration be divided into portions, each of which is the infinitely small extent necessary for a single fluctuation; and let the limits of any portion be β and $\beta + i$. For this portion we may put

$$\alpha = \beta + \iota, \tag{26 b}$$

and the corresponding integral, taken relatively to ι, is

$$\int_0^i N_{\beta+\iota} \quad f_{\beta+\iota}. \tag{27 b}$$

But by (533) of vol. 1,

$$f_{\beta+\iota} = f_\beta + \frac{\iota^n}{1.2....n} D^n f_\beta, \tag{28 b}$$

and, by definition,

$$\int_0^i N_{\beta+\iota} = 0. \tag{29 b}$$

Hence (27 b) becomes

$$f_\beta . \int_0^i N_{\beta+\iota} + \frac{D^n f_\beta}{1.2...n} \int_0^i . N_{\beta+\iota} \, \iota^n$$

$$= \frac{D^n f_\beta}{1.2...n} \int_0^i . N_{\beta+\iota} \, \iota^n. \tag{30 b}$$

But, by integrating by parts, we find

$$\int_0^i . N_{\beta+\iota} \, \iota^n = \frac{i^{n+1} N_{\beta+i} - \int_0^i \iota^{n+1} D_\iota N_{\beta+\iota}}{1+n}, \tag{31 b}$$

and the second member of (31 b) is, evidently, an infinitesimal

of the $(n+1)$st order, and (27 b) is, therefore, an infinitesimal of the same order. The number of all the portions of (25 b) is equal to $\frac{b-a}{i}$, and therefore the sum of all the portions (27 b) is an infinitesimal of the nth order; that is, this sum is infinitely small, and may be neglected, which gives at once the equation (25 b).

199. *Corollary.* If we take

$$\int' = \int_{-\infty}^{\infty}, \tag{32 b}$$

and if f_α is continuous and finite throughout its whole extent, (25 b) gives

$$\int_\alpha' \cdot N_\alpha f_\alpha = 0. \tag{33 b}$$

200. *Theorem. If the notation of* § 198 *is adopted, and if x is included between a and b, we shall have*

$$\int_a^b \cdot \frac{N_{\alpha-x} f_\alpha}{\alpha - x} = f_x \int_\alpha' \frac{N_\alpha}{\alpha}. \tag{34 b}$$

Proof. In the identical equation

$$\int_a^b = \int_a^{\alpha-i} + \int_{\alpha-i}^{\alpha+i} + \int_{\alpha+i}^b, \tag{35 b}$$

in which i is an infinitesimal, the first and third terms of the second member vanish by § 198, when this equation is substituted in the first member of (34 b). Hence if

$$\epsilon = \alpha - x, \tag{36 b}$$

we have

$$\int_a^b \cdot \frac{N_{\alpha-x} f_\alpha}{\alpha - x} = \int_{\alpha-i}^{\alpha+i} \frac{N_{\alpha-x} f_\alpha}{\alpha - x} = \int_{-i}^{+i} \frac{N_\epsilon f_{x+\epsilon}}{\epsilon}, \tag{37 b}$$

in the third member of which, the integrations are performed

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relatively to ι. But $f_{x+\iota}$ differs infinitely little from f_x, and, therefore, (37 b) gives

$$\int_a^b \cdot \frac{N_{\alpha-x} \cdot f_\alpha}{\alpha - x} = f_x \int_{-i}^{+i} \frac{N_\iota}{\iota}. \qquad (38\text{ b})$$

In the same way, when f_α is unity, and x is zero,

$$\int_\alpha' \frac{N_\alpha}{\alpha} = \int_{-i}^{+i} \frac{N_\iota}{\iota}, \qquad (39\text{ b})$$

which, substituted in (38 b), gives (34 b).

201. *Corollary.* Since we have

$$\int_\lambda' e^{\lambda(x-\alpha)\sqrt{-1}} = \frac{e^{\infty(\alpha-x)\sqrt{-1}} - e^{-\infty(\alpha-x)\sqrt{-1}}}{(\alpha - x)\sqrt{-1}}$$

$$= \frac{2 \sin. \infty (\alpha - x)}{\alpha - x}, \qquad (40\text{ b})$$

the first member of (40 b) may be substituted for $\frac{N_{\alpha-x}}{\alpha-x}$ in (34 b), which gives

$$\int_\alpha^b \cdot \int_\lambda' e^{\lambda(x-\alpha)\sqrt{-1}} f_\alpha = 2 f_x \int_\alpha' \frac{\sin. \infty \alpha}{\alpha}. \qquad (41\text{ b})$$

202. ***Problem.*** To find the value of

$$\int_\alpha' \frac{\sin. \infty \alpha}{\alpha}. \qquad (42\text{ b})$$

Solution. If we put

$$\beta \alpha = a, \qquad (43\text{ b})$$

we have

$$\int_\alpha' \frac{\sin. \beta \alpha}{\alpha} = \int_a' \frac{\sin. a}{a} = \int_\alpha' \frac{\sin. \alpha}{\alpha}; \qquad (44\text{ b})$$

that is, the first member of (44 b) is independent of the value

of β, as long as β is positive; so that if A is the required value of (42 b), we have

$$D_\beta A = 0. \tag{45 b}$$

We have also

$$\int_\alpha' \frac{\sin. \beta \alpha}{\alpha} = \int_\alpha' \frac{(1+\alpha^2) \sin. \beta \alpha}{\alpha (1+\alpha^2)}$$
$$= \int_\alpha' \frac{\sin. \beta \alpha}{\alpha (1+\alpha^2)} + \int_\alpha' \frac{\alpha \sin. \beta \alpha}{1+\alpha^2}. \tag{46 b}$$

Hence, by putting

$$B = \int_\alpha' \frac{\sin. \beta \alpha}{\alpha (1+\alpha^2)}, \tag{47 b}$$

we have

$$D_\beta B = \int_\alpha' \frac{\cos. \beta \alpha}{1+\alpha^2} \tag{48 b}$$

$$D_\beta^2 B = -\int_\alpha' \frac{\alpha \sin. \beta \alpha}{1+\alpha^2}; \tag{49 b}$$

whence, by (46 b),

$$A = B - D_\beta^2 B. \tag{50 b}$$

This equation may be regarded as a linear differential equation in which β is the independent variable, and its integral is

$$B = A + A' e^\beta + A'' e^{-\beta}, \tag{51 b}$$

in which A' and A'' are arbitrary constants. The values of these arbitrary constants may be determined from the extreme values of $D_\beta B$. When β is infinite, the value of (48 b) vanishes by (25 b); but (51 b) gives

$$D_\beta B = A' e^\beta - A'' e^{-\beta}, \tag{52 b}$$

which will not vanish, when β is infinite, unless

$$A' = 0; \tag{53 b}$$

whence

$$D_\beta B = -A'' e^{-\beta}. \tag{54 b}$$

Again when β is zero the value of (48 b) is π; and although β may never be supposed quite so small as zero, yet when it is an infinitesimal, (48 b) must differ infinitely little from π, and therefore, by (54 b),

$$\pi = -A'', \tag{55 b}$$

whence

$$D_\beta B = \pi e^{-\beta}. \tag{56 b}$$

Finally, the comparison of (47 b and 48 b) gives, by (56 b),

$$B = \int_0^\beta D_\beta B = \pi(1 - e^{-\beta}). \tag{57 b}$$

Hence, by (51 b),

$$A = \pi = \int_\alpha' \frac{\sin.\beta\alpha}{\alpha} = \int_\alpha' \frac{\sin.\infty\alpha}{\alpha}. \tag{58 b}$$

203. *Corollary.* The substitution of (58 b) in (41 b) gives

$$f_x = \frac{1}{2\pi}\int_a^b \cdot \int_\lambda' e^{\lambda(x-\alpha)\sqrt{-1}} f_\alpha, \tag{59 b}$$

provided the integral between the limits a and b is performed relatively to α.

204. *Corollary.* If f_α is the same as in (33 b), (59 b) gives

$$f_x = \frac{1}{2\pi}\int_\alpha' \int_\lambda' \cdot e^{\lambda(x-\alpha)\sqrt{-1}} f_\alpha = \frac{1}{2\pi}\int_{\alpha,\lambda}'^{2} e^{\lambda(x-\alpha)\sqrt{-1}} f_\alpha. \tag{60 b}$$

205. *Corollary.* In the same way, we should have

$$f_{x.\beta} = \frac{1}{2\pi}\int_\alpha' \int_\lambda' \cdot e^{\lambda(x-\alpha)\sqrt{-1}} f_{\alpha.\beta}, \tag{61 b}$$

$$f_{x.y} = \frac{1}{2\pi}\int_\beta' \int_\mu' \cdot e^{\mu(y-\beta)\sqrt{-1}} f_{x.\beta}; \tag{62 b}$$

whence, by substitution,

$$\begin{aligned} f_{x.y} &= \left(\frac{1}{2\pi}\right)^2 \int_\alpha' \int_\beta' \int_\lambda' \int_\mu' e^{[\lambda(x-\alpha)+\mu(y-\beta)]\sqrt{-1}} f_{\alpha.\beta} \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{\alpha.\beta.\lambda.\mu}'^{4} e^{[\lambda(x-\alpha)+\mu(y-\beta)]\sqrt{-1}} f_{\alpha.\beta}. \end{aligned} \tag{63 b}$$

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206. *Corollary.* In the same way,

$$f_{x.y.z} =$$
$$\left(\tfrac{1}{2}\pi\right)^3 \int'_\alpha \int''_\beta \int''_\gamma \int''_\lambda \int''_\mu \int''_\nu e^{[\lambda(x-\alpha)+\mu(y-\beta)+\nu(z-\gamma)]\sqrt{-1}} f_{\alpha.\beta.\gamma}$$
$$=\left(\tfrac{1}{2}\pi\right)^3 \int'^{6}_{\alpha.\beta.\gamma.\lambda.\mu.\nu} e^{[\lambda(x-\alpha)+\mu(y-\beta)+\nu(z-\gamma)]\sqrt{-1}} f_{\alpha.\beta.\gamma}. \quad (64\text{ b})$$

207. *Corollary.* The successive differentiation of (60 b) gives

$$D^n_x f_x = \tfrac{1}{2}\pi \int'^{2}_{\alpha.\lambda} (-1)^{\frac{n}{2}} \lambda^n e^{\lambda(x-\alpha)\sqrt{-1}} f_\alpha; \quad (65\text{ b})$$

and, in the same way, by the successive differentiation of (64 b), the factor $\lambda\sqrt{-1}$ is introduced under the signs of integration for each differentiation relatively to x, the factor $\mu\sqrt{-1}$ for each differentiation relatively to y, &c.

208. *Problem. To find several functions X_t, Y_t, &c. of the independent variables t, x, y, &c. which satisfy given linear differential equations with constant coefficients between various differential coefficients corresponding to the different independent variables, and which become given functions X_0, Y_0, &c. of the variable x, y, &c., when t becomes zero.*

Solution. Let $\mathfrak{X}_t$, $\mathfrak{Y}_t$ &c., $\mathfrak{X}_0$, $\mathfrak{Y}_0$, &c. represent the values of X_t, Y_t, &c., X_0, Y_0, &c. when x, y, &c. are changed into α, β, &c.; so that by (64 b) if n denotes the number of the variables x, y, &c.,

$$X_t = \left(\tfrac{1}{2}\pi\right)^n \int'^{2n}_{\alpha.\beta,\&c.\lambda.\mu,\&c.} e^{[\lambda(x-\alpha)+\mu(y-\beta)+\&c.]\sqrt{-1}} \mathfrak{X}_t \quad (66\text{ b})$$
&c.

If now $$R = L \quad (67\text{ b})$$

represents one of the given equations, in which R is a linear function of the differential coefficients with constant multipli-

ers, and L is a given function of t, x, y, &c.; and if $\mathfrak{L}$ denotes the value of L when x, y, &c. are changed to α, β, &c.; and $\mathfrak{R}$ the value of R when X_t, Y_t, &c. are changed to $\mathfrak{X}_t$, $\mathfrak{Y}_t$, &c., and D_x, D_y, &c. are changed to $\lambda\sqrt{-1}$, $\mu\sqrt{-1}$, &c.; the equation (67 b) is changed by the substitution of (66 b) into

$$\left(\tfrac{1}{2\pi}\right)^n \int_{\alpha\,.\,\beta,\ \&c.\ \lambda\,.\,\mu,\ \&c.}^{\prime\ 2n} e^{[\lambda(x-\alpha)+\mu(y-\beta)+\&c.]\sqrt{-1}}\,(\mathfrak{R}-\mathfrak{L})=0, \tag{68 b}$$

which is satisfied by putting

$$\mathfrak{R}=\mathfrak{L}, \tag{69 b}$$

and this equation involves no other differential coefficients than those taken relatively to t.

By this substitution, therefore, all the given equations are similarly transformed, and the problem is reduced to the integration of several linear differential equations with constant coefficients, in which there is only one independent variable; and this integration is performed by the method of § 179 to § 195. The functions to be determined are, in this new form, $\mathfrak{X}_t$, $\mathfrak{Y}_t$, &c., of which the initial values are $\mathfrak{X}_0$, $\mathfrak{Y}_0$, &c.

209. *Corollary.* It may be observed that for a complete solution, the initial values X'_0, Y'_0 &c. of some of the differential coefficients $D_t\,X_t$, $D_t\,Y_t$, &c. should also be given functions of x, y, &c.

210. Examples.

1. Integrate the equation

$$D_x^3\,X_t + D_t^3\,X_t = 0. \tag{70 b}$$

Solution. In this case, the substitution (66 b) gives

$$-\lambda^3\,\mathfrak{X}_t\sqrt{-1} + D_t^3\,\mathfrak{X}_t = 0,$$

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whence

$$X_t=\tfrac{1}{3}\left(e^{-\lambda t\sqrt{-1}}+e^{\frac{1}{2}(\sqrt{-1}-3)\lambda t}+e^{\frac{1}{2}(\sqrt{-1}+3)\lambda t}\right)X_0$$
$$+-\tfrac{1}{3\lambda}\left[e^{-\lambda t\sqrt{-1}}-\tfrac{1}{6}(1-\sqrt{-3})\,e^{\frac{1}{2}(\sqrt{-1}-3)\lambda t}\right.$$
$$\left.-\tfrac{1}{6}(1+\sqrt{-3})\,e^{\frac{1}{2}(\sqrt{-1}+3)\lambda t}\right]X_0'$$
$$+\tfrac{1}{3\lambda^2}\left[e^{\lambda t\sqrt{-1}}-\tfrac{1}{6}(1+\sqrt{-3})\,e^{\frac{1}{2}(\sqrt{-1}-3)\lambda t}\right.$$
$$\left.-\tfrac{1}{6}(1-\sqrt{-3})\,e^{\frac{1}{2}(\sqrt{-1}+3)\lambda t}\right]X_0''.$$

If the values of X_0, X_0' and X_0'' are written as follows,

$$X_0=f_\alpha,\quad X_0'=D_\alpha f_\alpha',\quad X_0''=D_\alpha^2 f_\alpha''; \qquad (71\text{ b})$$

we have, by (60 b and 65 b),

$$\textstyle\int_\alpha'\int_\lambda' X_0\, e^{\lambda(x-\alpha)\sqrt{-1}}\, e^{-\lambda t\sqrt{-1}}=\int_\alpha'\int_\lambda' X_0\, e^{\lambda(x-t-\alpha)\sqrt{-1}}=f_{x-t}$$
$$\textstyle\int_\alpha'\int_\lambda' X_0\, e^{\lambda(x-\alpha)\sqrt{-1}}\, e^{\frac{1}{2}(\sqrt{-1}\pm3)\lambda t}=f.\,_{x+\frac{1}{2}(1\mp\sqrt{-3})t},$$
$$\textstyle\int_\alpha'\int_\lambda' \frac{1}{\lambda} X_0'\, e^{\lambda(x-\alpha)\sqrt{-1}}\, e^{-\lambda t\sqrt{-1}}=f_{x-t}',$$
$$\textstyle\int_\alpha'\int_\lambda' \frac{1}{\lambda} X_0'\, e^{\lambda(x-\alpha)\sqrt{-1}}\, e^{\frac{1}{2}(\sqrt{-1}\pm3)\lambda t}=f.'_{x+\frac{1}{2}(1\mp\sqrt{-3})t},$$
$$\textstyle\int_\alpha'\int_\lambda' \frac{1}{\lambda^2} X_0''\, e^{\lambda(x-\alpha)\sqrt{-1}}\, e^{-\lambda t\sqrt{-1}}=f_{x-t}'',$$
$$\textstyle\int_\alpha'\int_\lambda' \frac{1}{\lambda^2} X_0''\, e^{\lambda(x-\alpha)\sqrt{-1}}\, e^{\frac{1}{2}(\sqrt{-1}\pm3)\lambda t}=f.''_{x+\frac{1}{2}(\mp\sqrt{-3})t};$$

whence we have

$$X_t=\tfrac{1}{3}\left(f_{x-t}+f.\,_{x+\frac{1}{2}(1-\sqrt{-3})t}+f.\,_{x+\frac{1}{2}(1+\sqrt{-3})t}\right)$$
$$-\tfrac{1}{3}\left(f'_{x-t}-\tfrac{1}{6}(1-\sqrt{-3})f.'_{x+\frac{1}{2}(1-\sqrt{-3})t}-\tfrac{1}{6}(1+\sqrt{-3})f'_{x+\frac{1}{2}(1+\sqrt{-3})t}\right)$$
$$+\tfrac{1}{3}\left(f''_{x-t}-\tfrac{1}{6}(1+\sqrt{-3})f.''_{x+\frac{1}{2}(1-\sqrt{-3})t}-\tfrac{1}{6}(1-\sqrt{-3})f.'_{x+\frac{1}{2}(1+\sqrt{-3})t}\right). \qquad (72\text{ b})$$

2. Integrate the equation

$$a\,b\,D_x^2 X_t+(a+b)\,D_{x\cdot t}^2 X_t+D_t^2 X_t=0. \qquad (73\text{ b})$$

Differential equations with constant coefficients.

Ans. With the notation of (71 b),

$$X_t = \frac{1}{a-b}\,(a\,f_{x-bt} - b\,f_{x-at} + f'_{x-bt} - f'_{x-at}). \quad (74\text{ b})$$

3. Integrate the equation

$$a^2\,D_x^2\,X_t + 2\,a\,D^2_{x\,.\,t}\,X_t + D_t^2\,X_t = 0. \quad (75\text{ b})$$

Ans. With the notation of (71 b),

$$X_t = f_{x-at} + t\,f'_{x-at} + a\,t\,D_x\,.f_{x-at}. \quad (76\text{ b})$$

4. Integrate the equation

$$a\,D_x\,X_t + D_t\,X_t = e^{mt+nx}. \quad (77\text{ b})$$

Solution. The value of $\mathfrak{X}_t$ in this case is

$$\mathfrak{X}_t = \mathfrak{X}_0\,e^{-a\lambda t\sqrt{-1}} + \frac{e^{mt+n\alpha} - e^{-at\lambda\sqrt{-1}+n\alpha}}{m + a\lambda\sqrt{-1}}; \quad (78\text{ b})$$

whence, by the notation of (71 b),

$$X_t = f_{x-at} - \frac{1}{2\pi}\int'^{\,2}_{\alpha\,.\,\lambda} e^{\lambda(x-\alpha)\sqrt{-1}}\,\frac{e^{mt+n\alpha} - e^{-at\lambda\sqrt{-1}+n\alpha}}{m + a\lambda\sqrt{-1}}. \quad (79\text{ b})$$

The value of the definite integral in (79 b) is found from the equation

$$\frac{1}{2\pi}\int'^{\,2}_{\alpha\,.\,\lambda} e^{\lambda(x-\alpha)\sqrt{-1}}\,e^{n\alpha} = e^{nx}, \quad (80\text{ b})$$

which, multiplied by $\frac{1}{a}\,e^{m'x}$ and integrated relatively to x gives

$$\frac{1}{2\pi}\int'^{\,2}_{\alpha\,.\,\lambda} \frac{e^{\lambda(x-\alpha)\sqrt{-1}+m'x}\;e^{n\alpha}}{am' + a\lambda\sqrt{-1}} = \frac{e^{nx+m'x}}{a\,m' + a\,n}; \quad (81\text{ b})$$

and this equation divided by $e^{m'x}$ is, by substituting m for $a\,m'$,

$$\frac{1}{2\pi}\int'^{\,2}_{\alpha\,.\,\lambda} \frac{e^{\lambda(x-\alpha)\sqrt{-1}}\;e^{n\alpha}}{m + a\lambda\sqrt{-1}} = \frac{e^{nx}}{m + a\,n}. \quad (82\text{ b})$$

Differential equations with constant coefficients.

The results, successively obtained from (82 b) by multiplying by e^{mt}, and again by substituting $x-at$ for x, reduce the value of the definite integral of (79 b) to

$$\frac{e^{mt+nx} - e^{n(x-at)}}{m+an}, \tag{83 b}$$

and the value of X_t is obtained by substituting (83 b) for the definite integral; so that

$$X_t = f_{x-at} + (83\,\text{b}). \tag{84 b}$$

Corollary. When $m = -an$, (85 b)

(83 b) is reduced to $t\,e^{n(x-at)}$. (86 b)

5. Integrate the equation

$$a D_x X_t + D_t X_t = t x. \tag{87 b}$$

Ans. $X_t = f_{x-at} + \frac{x^2 t}{2a^2} - \frac{1}{2} x^2 t + \frac{1}{2} a x t^2 - \frac{1}{6} a^2 t^3.$ (88 b)

6. Integrate the equation

$$D_x^2 X_t + D_t^2 X_t = (x^2 + t^2)\, e^{xt}. \tag{89 b}$$

Ans. $x = \frac{1}{2} f_{x+t\sqrt{-1}} + \frac{1}{2} f_{x-t\sqrt{-1}} + \frac{1}{2}(f'_{x-t\sqrt{-1}} - f'_{x+t\sqrt{-1}})\sqrt{-1} + e^{xt} - 1 - xt.$ (90 b)

7. Integrate the equation

$$a D_x X_t + b D_y X_t + D_t X_t = l\, e^{hx+ky+mt}. \tag{91 b}$$

Ans. $X_t = f_{x-at,\, y-bt} + \frac{l\, e^{hx+ky}\left[e^{mt} - e^{-(ha+kb)t}\right]}{ah+bk+m}$; (92 b)

which, when $m = -ah+bk$ (93 b)

is reduced to

$$X_t = f_{x-at,\, y-bt} + l t\, e^{hx+ky-(ha+kb)t}. \tag{94 b}$$

211. The integration of linear differential equations, in which the coefficients are not constant, can only be performed in some particular cases, some of which will be found in some of the following chapters.

CHAPTER XI.

INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

212. ***To integrate a given differential equation of the first order, between two variables x and t.***

Solution. Let t be the independent variable, and let the value of $D_t x$ be found from the given equation in the form

$$D_t x = -\frac{M}{N}. \tag{95 b}$$

The integral of this equation must involve an arbitrary constant a, from which the value of a can be found in terms of t and x in the form

$$a = A_t, \tag{96 b}$$

in which A_t is a function of t and x. The differential of (96 b) gives

$$0 = D_x A_t . D_t x + D_t A_t, \tag{97 b}$$

$$D_t x = -\frac{D_t A_t}{D_x A_t}. \tag{98 b}$$

Hence, by (95 b),

$$\frac{D_t A_t}{D_x A_t} = \frac{M}{N} = \frac{\lambda M}{\lambda N}, \tag{99 b}$$

in which λ is wholly arbitrary, and may, therefore, be taken of such a value that

$$\lambda N = D_x A_t, \tag{1 c}$$

which gives $\qquad \lambda M = D_t A_t;$ $\qquad$ (2 c)

whence, by the elimination of A_t,

$$D^2_{t \cdot x} A_t = D_t (\lambda N) = D_x (\lambda M). \qquad (3\,c)$$

There is no general process of finding a value of λ which will satisfy (3 c), and this problem must be solved in each case by the exercise of the ingenuity. When the value of λ is found, (1 c and 2 c) give

$$a = A_t = \int_t (\lambda M) = \int_x (\lambda N), \qquad (4\,c)$$

in which a is the arbitrary constant.

213. *Corollary*. An arbitrary function of x will be added to the third member of (4 c) to complete the integral, and an arbitrary function of t to the fourth member of (4 c). But these arbitrary functions are at once determined by the conditions that the third and fourth members are equal.

214. *Corollary*. The value of a is usually determined by the condition that x is to have a certain value x_τ, when t becomes τ. If, then, A_τ denotes the value of A_t when t and x are changed to τ and x_τ, (4 c) gives

$$A_t - A_\tau = 0. \qquad (5\,c)$$

215. *Corollary*. It is often the case that the given equation is such that it cannot be reduced to the form (95 b), and in this case the whole process must be left to the skill of the geometer.

216. *Corollary*. If M and N are such functions of x and t, that

$$M = M_x\, M_t\,, \qquad N = N_x\, N_t \qquad (6\,c)$$

in which M_x and N_x are functions of x alone, and M_t and N_t are functions of t alone, the value of λ may be assumed

$$\lambda = \frac{1}{M_x\, N_t}\,. \qquad (7\,c)$$

For this assumption reduces the two last members of (3 c) to zero. The equations (4 c and 5 c) give

$$0 = \int_{\tau}^{t} \frac{M_t}{N_t} + \int_{x_\tau}^{x} \frac{N_x}{M_x}. \tag{8 c}$$

217. *Corollary.* When M and N are homogenous functions of the same degree m, the value of λ is

$$\lambda = (N x + M t)^{-1}. \tag{9 c}$$

Hence $$\lambda^{-1} = N x + M t \tag{10 c}$$

$$D_x \lambda = -\lambda^2 (N + x D_x N + t D_x M) \tag{11 c}$$

$$D_t \lambda = -\lambda^2 (M + x D_t N + t D_t M) \tag{12 c}$$

$$D_x (\lambda M) = -\lambda^2 (M N + M x D_x N - N x D_x M) \tag{13 c}$$

$$D_t (\lambda N) = -\lambda^2 (M N - M t D_t N + N t D_t M). \tag{14 c}$$

But, by putting $$y = \frac{x}{t}$$

the expression $\frac{M}{t^m}$ becomes a function of y alone, which may be denoted by M', whence

$$D_x M' = D_y M' . D_x y = \frac{1}{t} D_y M' = \frac{D_x M}{t^m} \tag{15 c}$$

$$D_t M' = D_y M' . D_t y = -\frac{x}{t^2} D_y M'$$

$$= \frac{D_t M}{t^m} - \frac{m M}{t^{m+1}}, \tag{16 c}$$

and, therefore,

$$x D_x M = - t D_t M + m M; \tag{17 c}$$

and, in the same way,

$$x D_x N = - t D_t N + m N, \tag{18 c}$$

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which, substituted in (13 c), give by (14 c)

$$D_x(\lambda M) = -\lambda^2(MN - Mt\,D_t N + Nt\,D_t M) = D_t(\lambda N). \quad (19\text{ c})$$

Hence (3 c) is satisfied, and (4 c) gives

$$a = \int_t \frac{M}{Nx + Mt} = \int_x \frac{N}{Nx + Mt}. \quad (20\text{ c})$$

218. *Corollary.* If b is any function whatever of a, and if B_t is the same function of A_t, (96 b) gives

$$b = B_t. \quad (21\text{ c})$$

It may be shown, precisely as in § 212, that if μ is such that

$$\mu N = D_x B_t; \quad (22\text{ c})$$

μ will be a value of λ capable of satisfying the equation (3 c). If, however, b' is the differential coefficient of b taken relatively to a, and if B'_t is the same function of A_t which b' is of a, we have

$$D_x B_t = B'_t D_x A_t, \quad (23\text{ c})$$

whence (22 c and 1 c) give

$$\mu N = \lambda B'_t N \quad \text{or} \quad \mu = \lambda B'_t; \quad (24\text{ c})$$

that is, *the product of any value of λ by any function whatever of A_t is itself another value of λ.*

219. *Corollary.* Whenever M and N can be separated into such portions M', M'', M''', &c., and N', N'', N''', &c., that the equation (95 b) can be integrated when for M and N are substituted M' and N', or M'' or N'', &c., the integral of the equation itself is often readily obtained. For this purpose, let λ' and A'_t represent the values of λ and A_t which correspond to M' and N', λ'' and A''_t those which correspond to

M'' and N'', &c. it is necessary to find functions φ', φ'' &c. of A'_t, A''_t &c., which will satisfy the equation

$$\lambda' \varphi'. (A'_t) = \lambda'' \varphi''. (A''_t) = \lambda''' \varphi'''. (A'''_t) = \&c. \quad (25\,c)$$

For if the value of each member of (25 c) is denoted by λ, we shall have

$$\lambda M = \lambda' \varphi'. (A'_t) M' + \lambda'' \varphi'' (A''_t) M'' + \&c. \quad (26\,c)$$

$$\lambda N = \lambda' \varphi'. (A'_t) N' + \lambda'' \varphi'' (A''_t) N'' + \&c. \quad (27\,c)$$

But, by the preceding corollary,

$$D_t [\lambda' \varphi'. (A'_t) N'] = D_x [\lambda' \varphi' (A'_t) M], \&c. \quad (28\,c)$$

and therefore λ satisfies (3 c).

220. Examples.

1. Integrate the equation

$$(t X' + T) D_t x + X + x T' = 0, \quad (29\,c)$$

in which X is a given function of x, and X' its differential coefficient; and T is a given function of t, and T' its differential coefficient.

Solution. In this case,

$$M = X + x T'$$

$$N = T + t X'$$

$$D_t N = T' + X' = D_x M,$$

and, therefore, (3 c) is satisfied by

$$\lambda = 1.$$

Hence the required integral is

$$a = \textstyle\int_t (X + x T') = \int_x (T + t X')$$

$$= X t + x T; \quad (30\,c)$$

Equations of the first order.

or if $\mathfrak{X}$ and $\mathfrak{T}$ are the values of X and T when t and x are changed to τ and x_τ,

$$\mathfrak{X}\,\tau + x_\tau\,\mathfrak{T} = X\,t + x\,T. \tag{31 c}$$

2. Integrate the equation

$$(t\cos.x + \sin.t)\,D_t\,x + \sin.x + x\cos.t = 0. \tag{32 c}$$

Ans. $\tau\sin.x_\tau + x_\tau\sin.\tau = t\sin.x + x\sin.t.$ (33 c)

3. Integrate the equation

$$x^a\,t^b\,D_t\,x + k\,x^{a'}\,t^{b'} = 0. \tag{34 c}$$

Ans. $\dfrac{x^{a-a'+1} - x_\tau^{a-a'+1}}{a-a'+1} + k.\dfrac{t^{b'-b+1} - \tau^{b'-b+1}}{b'-b+1}.$ (35 c)

Corollary. When $a - a' + 1 = 0$,
the answer is

$$\log.\frac{x}{x_\tau} + k\,\frac{t^{b'-b+1} - \tau^{b'-b+1}}{b'-b+1}; \tag{36 c}$$

when $b' - b + 1 = 0$,
it is

$$\frac{x^{a-a'+1} - x_\tau^{a-a'+1}}{a-a'+1} + k\log.\frac{t}{\tau} = 0; \tag{37 c}$$

and when both these conditions are satisfied, it is

$$x\,\tau^k + t\,x_\tau^k = 0. \tag{38 c}$$

4. Integrate the equation

$$t\,D_t\,x = x + \sqrt{(x^2+t^2)}. \tag{39 c}$$

Solution. This is a homogeneous equation, and (9 c) gives

$$\lambda^{-1} = M\,t + N\,x = t\,x - t\,x + t\sqrt{(x^2+t^2)} = t\sqrt{(x^2+t^2)};$$

hence, by (20 c), the integral is

$$a = -\textstyle\int_x (x^2+t^2)^{-\frac{1}{2}} = \int_t x\,t^{-1}(x^2+t^2)^{-\frac{1}{2}} + t^{-1}$$
$$= \log.[\sqrt{(x^2+t^2)} - x], \tag{40 c}$$

Equations of the first order.

or $$\sqrt{(x^2+t^2)} - x = \sqrt{(x_\tau^2+\tau^2)} - x_\tau, \tag{41 c}$$

or $$t^2 = a^2 + 2\,a\,x, \tag{42 c}$$

in which a is the arbitrary constant.

5. Integrate the equation

$$D_t\,x = \frac{x}{t} + \frac{x}{t}\log.\frac{x}{t}. \tag{43 c}$$

$$Ans.\quad \left(\frac{x}{t}\right)^{\tau} = \left(\frac{x_\tau}{\tau}\right)^{t}. \tag{44 c}$$

6. Integrate the equation

$$D_t x = \frac{x}{t} - \frac{n\,x}{t}\log.\frac{x}{t}. \tag{45 c}$$

$$Ans.\quad \left(\frac{x}{t}\right)^{t^n} = \left(\frac{x_\tau}{\tau}\right)^{\tau^n}. \tag{46 c}$$

7. Integrate the equation

$$\left(1 + m\log.\frac{x}{t}\right)D_t x = \frac{x}{t}\left(1 + n\log.\frac{x}{t}\right). \tag{47 c}$$

$$Ans.\quad \left(\frac{x}{t}\right)^{x^m\,t^n} = \left(\frac{x_\tau}{\tau}\right)^{x_\tau^m\,\tau^n}. \tag{48 c}$$

8. Integrate the equation

$$(h\,t^m\,x^{n+1} + h'\,t^{m'}\,x^{n'+1})$$
$$+(k\,t^{m+1}\,x^n + k'\,t^{m'+1}\,x^{n'})\,D_t\,x = 0. \tag{49 c}$$

Solution. This is a case of § 219, and by putting

$$\left.\begin{aligned} M' &= h\,t^m\,x^{n+1}, & M'' &= h'\,t^{m'}\,x^{n'+1},\\ M' &= k\,t^{m+1}\,x^n, & M'' &= k'\,t^{m'+1}\,x^{n'}; \end{aligned}\right\} \tag{50 c}$$

we have

$$\left.\begin{aligned} \lambda^{\prime-1} &= t^{m+1}\,x^{n+1}, & \lambda^{\prime\prime-1} &= t^{m'+1}\,x^{n'+1},\\ A_t' &= \log.\,t^h\,x^k, & A_t'' &= \log.\,t^{h'}\,x^{k'}, \end{aligned}\right\} \tag{51 c}$$

Equations of the first order.

and if α and α' are taken to satisfy the equation

$$t^{m+1}\,x^{n+1}\,(t^h\,x^k)^\alpha = t^{m'+1}\,x^{n'+1}\,(t^{h'}\,x^{k'})^{\alpha'},$$

which gives

$$\alpha\,h + m = \alpha'\,h' + m', \quad \alpha\,k + n = \alpha'\,k' + n' \tag{53 c}$$

$$\alpha = \frac{(m - m')\,k' - (n - n')\,h'}{h'\,k - h\,k'} \tag{54 c}$$

$$\alpha' = \frac{(m - m')\,k - (n - n')\,h}{h'\,k - h\,k'}, \tag{55 c}$$

we may put

$$\lambda^{-1} = t^{\alpha\,h + m + 1}\,x^{\alpha\,k + n + 1}, \tag{56 c}$$

and the integral of (49 c) becomes

$$\frac{1}{\alpha\,(t^h\,x^k)^\alpha} + \frac{1}{\alpha'(t^{h'}x^{k'})^{\alpha'}} = \frac{1}{\alpha\,(\tau^h\,x_\tau^k)^\alpha} + \frac{1}{\alpha'\,(\tau^{h'}\,x_\tau^{k'})^{\alpha'}}. \tag{57 c}$$

9. Integrate the equation

$$(3\,a\,x\,t + 2\,b\,t^2)\,D_t\,x + 2\,a\,x^2 + 3\,b\,x\,t = 0.$$

Ans. $a\,x^3\,t^2 + b\,t^3\,x^2 = a\,x_\tau^3\,\tau^2 + b\,x_\tau^2\,\tau^3.$

10. Integrate the equation

$$(3\,a\,x^3\,t^3 + 2\,b\,t)\,D_t\,x + 2\,a\,x^4\,t^2 + 3\,b\,x = 0.$$

Ans. $a\,(x^3\,t^2 - x_\tau^3\,\tau^2) + b\log.\dfrac{x^2\,t^3}{x_\tau^2\,\tau^3} = 0.$

11. Integrate the equation

$$(h\,x + k\,t + a)\,D_t\,x + h'\,x + k'\,t + a' = 0. \tag{58 c}$$

Solution. Put, in this equation,

$$x = x' + \beta, \quad t = t' + \alpha, \tag{59 c}$$

and we have $\quad D_t\,x = D_t\,x' = D_{t'}\,x',$ (60 c)

whence (58 c) gives (61 c)

$$(h\,x' + k\,t' + h\,\beta + k\,\alpha + a)\,D_{t'}\,x' + h'\,x' + k'\,t' + h'\,\beta + k'\,\alpha + a' = 0;$$

and if α and β are taken such that

$$h\beta + k\alpha + a = 0, \quad h'\beta + k'\alpha + a' = 0, \tag{62 c}$$

(61 c) becomes

$$(h x' + k t')\, D_{t'}\, x' + h' x' + k' t' = 0, \tag{63 c}$$

which may be integrated like any other homogeneous equation.

12. Integrate the equation

$$D_t\, x + T x = T', \tag{64 c}$$

in which T and T' are functions of t.

Solution. Let

$$t' = \int_t T, \tag{65 c}$$

whence

$$D_t\, t' = T \tag{66 c}$$

$$D_t\, x = D_{t'}\, x\, D_t\, t' = T\, D_{t'}\, x; \tag{67 c}$$

and if T'' denotes the value of $\frac{T'}{T}$ when t' is substituted for t, (64 c) gives

$$D_{t'}\, x + x = T'', \tag{68 c}$$

which may be integrated by the processes of the preceding chapter, since it is linear, with constant coefficients. The integral is, if τ' is the value of t' when t becomes τ,

$$\begin{aligned} x &= x_\tau\, e^{\tau' - t'} + e^{-t'} \int_{\tau'}^{t'} T''\, e^{t'} \\ &= x_\tau\, e^{-\int_\tau^t T} + e^{-\int T} \int_\tau^t T'\, e^{\int T}. \end{aligned} \tag{69 c}$$

13. Integrate the equation

$$D_t\, x + \frac{k x}{t + h} = \frac{a\,(t + h')^{k'}}{(t + h)^k}.$$

Ans. $x = x_\tau \frac{(\tau + h)^k}{(t + h)^k} + \frac{a\,(t + h')^{k'+1} - a\,(\tau + h')^{k'+1}}{(k'+1)\,(t + h)^k}.$

14. Integrate the equation

$$D_t\, x + h\, x^2 = k\, t^m, \tag{70 c}$$

which is called *Riccati's equation.*

Riccati's equation.

Solution. Let x' and t' be so taken that

$$x = \frac{1}{h\,t} + \frac{1}{x'\,t^2}, \quad t' = t^{m+3}; \qquad (71\text{ c})$$

and we have

$$D_t x' = D_{t'} x' . D_t t' = (m+3) t^{m+2} D_{t'} x' = (m+3)\frac{t'}{t} D_{t'} x' \qquad (72\text{ c})$$

$$D_t x = -\frac{1}{h\,t^2} - \frac{2}{x'\,t^3} - \frac{D_t\,x'}{x'^2\,t^2}$$

$$= -\frac{1}{h\,t^2} - \frac{2}{x'\,t^3} - (m+3)\frac{t'\,D_{t'}\,x'}{x'^2\,t^3} \qquad (73\text{ c})$$

$$h\,x^2 = \frac{1}{h\,t^2} + \frac{2}{x'\,t^3} + \frac{h}{x'^2\,t^4} \qquad (74\text{ c})$$

$$D_t\,x + h\,x^2 = \frac{h-(m+3)t\,t'\,D_{t'}\,x'}{x'^2\,t^4} = \frac{k\,x'^2\,t'\,t}{x'^2\,t^4}. \qquad (76\text{ c})$$

Hence

$$D_{t'}\,x' + \frac{k}{m+3}\,x'^2 = \frac{h}{m+3}\cdot\frac{1}{t\,t'} = \frac{h}{m+3}\,t'^{-\frac{m+4}{m+3}}. \qquad (77\text{ c})$$

And if
$$k' = \frac{h}{m+3}, \quad h' = \frac{k}{m+3}, \quad m' = -\frac{m+4}{m+3} \qquad (78\text{ c})$$

$$D_{t'}\,x' + h'\,x'^2 = k'\,t'^{m'}, \qquad (79\text{ c})$$

which is of the same form with the given equation. Hence if Riccati's equation can be integrated for any value m' of m, it can also be integrated for the value m determined by (78 c); and if it can be integrated for the value m, it can also be integrated for the value m'.

Let i be determined, so that,

$$m = -\frac{4\,i}{2\,i+1}, \qquad (80\text{ c})$$

and (78 c) gives

$$m' = -\frac{4i+4}{2i+3} = -\frac{4(i+1)}{2(i+1)+1}; \tag{81 c}$$

so that m' is obtained from m by increasing i by unity. Hence if the equation can be integrated for any value of i, it can also be integrated for the values of i, which are greater or less by unity, and therefore for any value which differs from i by any integer whatever.

But when $$i = 0, \tag{82 c}$$
we have $$m = 0, \tag{83 c}$$
and Riccati's equation becomes

$$D_t\, x + h\, x^2 = k, \tag{84 c}$$

the integral of which is

$$t - \tau = \int_{x_\tau}^{x} \frac{1}{k - h\, x^2} = \frac{1}{\sqrt{h\, k}} \log. \frac{(\sqrt{k} + x\sqrt{h})\sqrt{(k - h\, x_\tau^2)}}{\sqrt{(k - h\, x^2)}(\sqrt{k} + x_\tau\sqrt{h})}; \tag{85 c}$$

so that Riccati's equation may be integrated whenever i is an integer either positive or negative.

When $$i = \pm \infty, \tag{86 c}$$
we have $$m = -2, \tag{87 c}$$
and therefore this case would only be obtained from the preceding, by an infinite succession of substitutions. This case, however, admits of direct integration, for, by the substitution

$$x = \frac{1}{h\, t} + \frac{x'}{t^2}, \tag{88 c}$$

Riccati's equation becomes in this case

$$t^2\, D_t\, x' + x'^2 = k\, t^2, \tag{89 c}$$

which is homogeneous, and its integral is

$$\frac{[2\, x + t - t\sqrt{(1+4k)}][2\, x_\tau + \tau + \tau\sqrt{(1+4k)}]}{[2\, x + t + t\sqrt{(1+4\, k)}][2\, x_\tau + \tau - \tau\sqrt{(1+4\, k)}]} = \left(\frac{\tau}{t}\right)^{\sqrt{(1+4k)}} \tag{90 c}$$

15. Integrate the equation

$$P = 0, \tag{91 c}$$

in which P is a given function of $D_t\, x$.

Solution. By solving the equation (91 c) relatively to $D_t\, x$, each of its values will be found to be a constant, one of which we may denote by m.

Hence
$$D_t\, x = m, \tag{92 c}$$
whence
$$x - x_\tau = m\,(t - \tau) \tag{93 c}$$
and
$$m = \frac{x - x_\tau}{t - \tau} = D_t\, x, \tag{94 c}$$

and the second member of (94 c) may therefore be substituted for $D_t\, x$ in (91 c); and if Q represents the value of P arising from this substitution, the integral of (91 c) is

$$Q = 0. \tag{95 c}$$

16. Integrate the equation

$$D_t\, x^2 = a^2.$$

Ans. $(x - x_\tau)^2 = a^2 (t - \tau)^2$.

17. Integrate the equation

$$\sqrt{(1 + D_t\, x^2)} = a + b\, D_t\, x.$$

Ans. $\sqrt{[(x - x_\tau)^2 + (t - \tau)^2]} = a(t - \tau) + b(x - x_\tau)$.

18. Integrate the equation

$$D_t\, x^n = T, \tag{96 c}$$

in which T is a function of t.

Ans. $x_t - x_\tau = \int_\tau^t \sqrt[n]{T}, \quad (97\text{ c})$

or the equation which is obtained by freeing (97 c) from radicals.

19. Integrate the equation

$$D_t\, x^3 = t.$$

Ans. $\left(x - x_\tau + \tau^{\frac{4}{3}}\right)^3 = \frac{27}{64}\, t^4.$

20. Integrate the equation

$$P = t, \tag{98 c}$$

in which P is a given function of $D_t\, x$.

Solution. By putting $p = D_t\, x,$ (99 c)
(262) gives

$$\begin{aligned} x &= \textstyle\int_t D_t\, x = \int_t p = \int_t p\, D_t t = p\, t - \int_t t\, D_t\, p \\ &= p\, t - \textstyle\int_p t\, D_p\, p = p\, t - \int_p . P, \end{aligned} \tag{1 d}$$

and the integral is obtained by eliminating p between (1 d) and the equation obtained from (98 c) by changing $D_t\, x$ to p.

21. Integrate the equation

$$t = D_t\, x + e^{D_t\, x} + \sin.\, D_t\, x.$$

Ans. It is the equation obtained by eliminating p and p_τ between the equations

$$\begin{aligned} t &= p + e^p + \sin p \\ \tau &= p_\tau + e^{p_\tau} + \sin.\, p_\tau \\ x - x_\tau &= p\, t - p_\tau \tau - \tfrac{1}{2}\,(p^2 - p_\tau^2) - e^p + e^{p_\tau} + \cos.\, p - \cos.\, p_\tau\,. \end{aligned}$$

22. Integrate the equation

$$t + a\, D_t\, x = b \sqrt{(1 + D_t\, x^2)}.$$

Ans. It is the equation obtained by eliminating p and p_τ between the equations

$$\begin{aligned} t + a\, p &= b \sqrt{(1 + p^2)} \\ \tau + a\, p_\tau &= b \sqrt{(1 + p_\tau^2)} \\ x - x_\tau &= p\, t - p_\tau \tau + \tfrac{1}{2}\, a\, (p^2 - p_\tau^2) - \tfrac{1}{2}\, p \sqrt{(1 + p^2)} \\ &\quad + \tfrac{1}{2}\, p_\tau \sqrt{(1 + p_\tau^2)} + \tfrac{1}{2} \log.\, \frac{p + \sqrt{(1 + p^2)}}{p_\tau + \sqrt{(1 + p_\tau^2)}}\,. \end{aligned}$$

Homogeneous equations of first order.

23. Integrate the differential equation of the first degree, which is homogeneous in reference to the variables x and t.

Solution. Let
$$y = \frac{x}{t}, \quad D_t x = p, \tag{2 d}$$
which gives
$$x = y\,t \tag{3 d}$$
$$D_y x = y\,D_y t + t = D_t x\,D_y t = p\,D_y t \tag{4 d}$$
$$\frac{D_y t}{t} = \frac{1}{p-y} \tag{5 d}$$
$$\log. t = \int_y \frac{1}{p-y}. \tag{6 d}$$

But the substitution of (2 d) in the given equation reduces to an equation containing only p and y; hence the integral (6 d) is readily obtained, and the required integral is obtained by eliminating p and y from (3 d, 6 d) and the given equation in the form to which it is reduced by the substitution of (2 d).

24. Integrate the equation
$$D_t x = \frac{x}{t} - n\sqrt{(1 + D_t x^2)}.$$

Ans. The equation resulting from the elimination of p and p_τ between the equations
$$x = p\,t + n\,t\sqrt{(1+p^2)}$$
$$x_\tau = p_\tau \tau + n\,\tau\sqrt{(1+p_\tau^2)}$$
$$\frac{x}{x_\tau}\sqrt{\left(\frac{1+p^2}{1+p_\tau^2}\right)} = \left(\frac{p_\tau + \sqrt{(1+p_\tau^2)}}{p + \sqrt{(1+p^2)}}\right)^{\frac{1}{n}}.$$

25. Integrate the equation
$$x = P\,t + Q, \tag{7 d}$$
in which P and Q are functions of $D_t x$.

Equations of the first order.

Solution. Let $$p = D_t\, x, \tag{8 d}$$

and the differential of (7 d) gives

$$\begin{aligned} D_p\, x &= D_t\, x \,.\, D_p\, t = p\, D_p\, t \\ &= P\, D_p t + t\, D_p\, P + D_p\, Q, \end{aligned} \tag{9 d}$$

or $$(p - P)\, D_p t - t\, D_p\, P = D_p\, Q, \tag{10 d}$$

which is a linear equation of the first order, by taking p as the independent variable. The integral of (10 d) is an equation between t and p from which p can be eliminated by means of the given equation.

26. Integrate the equation

$$x = (D_t x - \frac{1}{m})\, t + e^{m\, D_t\, x}.$$

Ans. The integral is found by eliminating p and p_τ between the equations

$$x = (p - \tfrac{1}{m})\, t + e^{mp}$$

$$x_\tau = (p_\tau - \tfrac{1}{m})\, \tau + e^{m\, p_\tau}$$

$$t - m\, (p - p_\tau) e^{m\, p} = \tau\, e^{m(p - p_\tau)}.$$

27. Integrate the equation

$$x = t\, D_t\, x + P, \tag{11 d}$$

in which P is a function of $D_t\, x$.

Ans. If P_τ denotes the value which P obtains when $\frac{x - x_\tau}{t - \tau}$ is substituted for $D_t\, x$, the required integral is

$$t\, x_\tau - x\tau = (t - \tau)\, P_\tau\,. \tag{12 d}$$

28. Integrate the equation

$$x = t\,D_t x + n\sqrt{(1 + D_t x^2)}.$$

Ans. $t\,x_\tau - x\,\tau = n\sqrt{[(t-\tau)^2 + (x - x_\tau)^2]}$,

or $(t\,x_\tau - x\tau)^2 = n^2(t-\tau)^2 + n^2(x - x_\tau)^2$.

29. Integrate the equation

$$D_t x = (A\,t^a + B\,x^b)^{\frac{1}{b} - \frac{1}{a}}. \tag{13 d}$$

Solution. Let $u = x\,t^{-\frac{a}{b}}$,

or $x = u\,t^{\frac{a}{b}}$;

whence $D_t x = t^{\frac{a}{b}}\,D_t u + \frac{a}{b}\,u\,t^{\frac{a-b}{b}}$

$$= (A + B\,u^b)^{\frac{1}{b} - \frac{1}{a}}\,t^{\frac{a-b}{b}},$$

and

$$\frac{1}{t} = \frac{b\,D_t u}{b\,(A + B\,u^b)^{\frac{1}{b} - \frac{1}{a}} - a\,u}$$

$$\log.\frac{t}{\tau} = \int_{u_\tau}^{u} \cdot \frac{b}{b\,(A + B\,u^b)^{\frac{1}{b} - \frac{1}{a}} - a\,u}, \tag{14 d}$$

in which $u_\tau = x_\tau\,\tau^{-\frac{a}{b}}$. (15 d)

30. Integrate the equation

$$\frac{t\,D_t x - x}{\sqrt{(x^2 + t^2)}\sqrt{(D_t x^2 + 1)}} = \left[\left(\frac{f.\frac{x}{t}}{F.(x^2 + t^2)}\right)^2 + 1\right]^{-\frac{1}{2}} \tag{16 d}$$

in which $f.$ and $F.$ are any given functions.

Solution. Let r and φ be taken so that

$$t = r \cos. \varphi, \quad x = r \sin. \varphi, \tag{17 d}$$

whence

$$r^2 = x^2 + t^2, \quad \tan. \varphi = \frac{x}{t} \tag{18 d}$$

$$\left.\begin{aligned} D_t x &= \sin. \varphi\, D_t r + r \cos. \varphi\, D_t \varphi \\ 1 &= \cos. \varphi\, D_t r - r \sin. \varphi\, D_t \varphi \end{aligned}\right\} \tag{19 d}$$

$$\left.\begin{aligned} t\, D_t x - x &= r^2\, D_t \varphi \\ D_t x^2 + 1 &= D_t r^2 + r^2\, D_t \varphi^2, \end{aligned}\right\} \tag{20 d}$$

and (16 d) becomes

$$r D_t \varphi \sqrt{\left[\left(\frac{f. \tan. \varphi}{F. r^2}\right)^2 + 1\right]} = \sqrt{(D_t r^2 + r^2 D_t \varphi^2)}, \tag{21 d}$$

or

$$f. \tan. \varphi \,.\, D_t \varphi = r^{-1} D_t r \,.\, F. r^2, \tag{22 d}$$

and its integral is

$$\int_{\varphi_\tau}^{\varphi} f. \tan. \varphi = \int_{r_\tau}^{r} \frac{F. r^2}{r}, \tag{23 d}$$

in which

$$\tan. \varphi_\tau = \frac{x_\tau}{\tau}, \quad r_\tau^2 = x_\tau^2 + \tau^2. \tag{24 d}$$

31. Integrate the equation

$$\frac{t\, D_t x - x}{\sqrt{(D_t x^2 + 1)}} = f. (x^2 + t^2).$$

Ans. By the notation of the preceding example,

$$\varphi - \varphi_\tau = \int_{r_\tau}^{r} \frac{f. r^2}{r\sqrt{[r^2 - (f. r^2)^2]}}.$$

32. Integrate the equation

$$\frac{t\, D_t x - x}{\sqrt{(x^2 + t^2)} . \sqrt{(D_t x^2 + 1)}} = f. \frac{x}{t}.$$

Ans. By the notation of example 30,

$$\int_{\varphi_\tau}^{\varphi} \frac{\sqrt{[1-(f.\tan.\varphi)^2]}}{f.\tan.\varphi} = \log.\frac{r}{r_\tau}.$$

33. Integrate the equation

$$\frac{t\,D_t x - x}{\sqrt{(t^2-x^2)}\sqrt{(1-D_t x^2)}} = \left[\left(\frac{f.\frac{x}{t}}{F(t^2-x^2)}\right)^2 - 1\right]^{-\frac{1}{2}}. \quad (25\,\text{d})$$

Ans. By putting

$$\left.\begin{aligned} r^2 &= t^2 - x^2, \quad \text{Tan.}\,\varphi = \frac{x}{t} \\ r_\tau^2 &= \tau^2 - x_\tau^2 \quad \text{Tan.}\,\varphi = \frac{x_\tau}{\tau}, \end{aligned}\right\} \quad (26\,\text{d})$$

the integral is

$$\int_{\varphi_\tau}^{\varphi} . f.\,\text{Tan.}\,\varphi = \int_{r_\tau}^{r} . \frac{F.\,r^2}{r}. \quad (27\,\text{d})$$

34. Integrate the equation

$$\frac{t\,D_t x + x}{\sqrt{(x^2-t^2)}\sqrt{(D_t x^2-1)}} = \left[\left(\frac{f.\,(x\,t)}{F.\,(x^2+t^2)}\right)^2 - 1\right]^{-\frac{1}{2}}. \quad (28\,\text{d})$$

Ans. By putting

$$\left.\begin{aligned} r^2 &= x^2 + t^2, \quad p = x\,t, \\ r_\tau^2 &= x_\tau^2 + \tau^2, \quad p_\tau = x_\tau\,\tau; \end{aligned}\right\} \quad (29\,\text{d})$$

the integral is

$$\int_{r_\tau}^{r} r\,F.\,r^2 = \int_{p_\tau}^{p} f.p. \quad (30\,\text{d})$$

221. *Problem. To integrate several differential equations between several variables and their differential coefficients taken with respect to one of them regarded as the independent variable.*

Solution. By taking the successive differentials of these equations with respect to the independent variable, as many new equations may be obtained as may be necessary to eliminate from them, combined with the given equations, all the variables but two, of which one is the independent variable, together with their differential coefficients. The resulting equation will be an equation between these two variables, and the successive differential coefficients of one variable taken with respect to the other, which is the independent variable.

In most cases, however, the integration can only be obtained by some ingenious device. Examples of this problem will occur under the subsequent problem.

222. *Problem. To find a function* ψ *of several independent variables* t, x, y, *&c., which satisfies a given differential equation of the first order, and becomes a given function of the variables* x, y, *&c., for a given value* τ *of the variable* t.

Solution. If D' denotes the differential coefficient with reference to the given function of the variables x, y, &c., and if s, p, q, &c. denote the differential coefficients $D_t \psi$, $D_x \psi$, $D_y \psi$, &c., we have

$$D' \psi = p\, D' x + q\, D' y + \&c. \tag{31 d}$$

If, moreover, x, y, &c. instead of being independent of t, were assumed to be certain functions of t, we should have

$$D_t \psi = s + p\, D_t x + q\, D_t y + \&c., \tag{32 d}$$

the differential coefficient of which, relatively to D' is

$$\begin{aligned} D' D_t \psi = p\, D' D_t x &+ q\, D' D_t y + \&c. \\ &+ D' s + D_t x \,.\, D' p + D_t y \,.\, D' q + \&c. \end{aligned} \tag{33 d}$$

Equations of the first order.

But the differential coefficient of (31 d) relatively to t is, by this assumption,

$$D' D_t \psi = p\, D' D_t x + q\, D' D_t y + \&c.$$
$$+ D_t p \,.\, D' x + D_t q \,.\, D' y + \&c.\,; \quad (34\,d)$$

and the difference between (33 d and 34 d) is

$$0 = D' s + D_t x \,.\, D' p + D_t y \,.\, D' q + \&c.$$
$$- D_t p \,.\, D' x - D_t q \,.\, D' y + \&c. \quad (35\,d)$$

If the given differential equation becomes by the substitution of s, p, q, &c. for the differential coefficients of ψ,

$$R = 0, \quad (36\,d)$$

its differential coefficient is

$$0 = D_\psi R \,.\, D' \psi + D_s R \,.\, D' s + D_p R \,.\, D' p + D_q R \,.\, D' q + \&c.$$
$$+ D_x R \,.\, D' x + D_y R \,.\, D' y + \&c., \quad (37\,d)$$

which becomes, by the substitution of (31 d),

$$0 = D_s R \,.\, D' s + D_p R \,.\, D' p + D_q R \,.\, D' q + \&c.$$
$$+ (p\, D_\psi R + D_x R) D' x + (q\, D_\psi R + D_y R) D' y + \&c. \quad (38\,d)$$

But (36 d) is the only given equation between the quantities t, x, y, &c., s, p, q, &c., and cannot, therefore, determine more than one of the differential coefficients $D' s$, &c. in terms of the others; so that the value of this differential coefficient, determined from (38 d), must be the same with that given by (35 d); and, consequently, the product of (35 d) by $D_s R$ must coincide with (38 d). Hence

$$\frac{1}{D_s R} = \frac{D_t x}{D_p R} = \frac{D_t y}{D_q R} = \&c.$$
$$= \frac{-D_t p}{p\, D_\psi R + D_x R} = \frac{-D_t q}{q\, D_\psi R + D_y R} = \&c.\,; \quad (39\,d)$$

and (32 d) gives, by the theory of proportions, each of these fractions, equal to

$$\frac{D_t\,\psi}{s\,D_s\,R + p\,D_p\,R + q\,D_q\,R + \&c.}\,. \tag{40 d}$$

The equations (36 d, 39 d, and 40 d) may then be regarded as several equations of the first order, with one independent variable, and the values of x, y, &c., p, q, &c., ψ may be determined in terms of t and of their values x_τ, y^τ &c., p_τ, q_τ &c., ψ_τ corresponding to the value τ of t.

Since for the value τ of t, ψ becomes a given function of x, y, &c., it is evident that ψ_τ must be the same function of x_τ, y_τ, &c.; and also that p_τ, q_τ, &c. must be the differential coefficients of ψ_τ with reference to x_τ, y_τ, &c.

If from the integrals of (39 d and 40 d) the quantities x_τ, y_τ &c. are all eliminated, and the value of ψ obtained, this value is evidently such a function of t, x, y, &c. that if t, x, y, &c. are changed to τ, x_τ, y_τ, &c., ψ will become ψ_τ; but by the simple change of t to τ, ψ must become the same function of x, y, &c. which ψ_τ is of x_τ, y_τ, &c.; that is, the value of ψ obtained by this process of elimination satisfies the problem.

223. Examples.

1. *Integrate the linear differential equation of the first order, involving any number of independent variables.*

Solution. This equation may be written in the form

$$T\,D_t\,\psi + X\,D_x\,\psi + Y\,D_y\,\psi + \&c. = M, \tag{41 d}$$

in which T, X, Y, &c., M, are functions of t, x, y, &c. and ψ.

In this case (36 d) becomes

$$R = T s + X p + Y q + \&c. - M = 0, \qquad (42\,d)$$

whence

$$D_s R = T, \qquad D_p R = X, \&c. \qquad (43\,d)$$

$$s\, D_s R + p\, D_p R + \&c. = T s + X p + \&c. = M, \qquad (44\,d)$$

and (39 d and 40 d) become

$$\frac{1}{T} = \frac{D_t x}{X} = \frac{D_t y}{Y} = \&c. = \frac{D_t \psi}{M}$$

$$= \frac{-D_t p}{p\, D_\psi R + D_x R} = \frac{-D_t q}{q\, D_\psi R + D_y R} = \&c. \qquad (45\,d)$$

The fractions in the first line of (45 d) do not involve p, q, &c., and therefore the integrals of the equations in this line give the required value of ψ, without resorting at all to the second line of (45 d).

Corollary. Whenever, by the combination of the equations in the first line of (45 d), a number of equations is found equal to that of the variables ψ, x, y, &c., and admitting of direct integration, such as

$$D_t U = 0, \quad D_t V = 0, \&c., \qquad (46\,d)$$

the required integrals are

$$U = U_\tau, \quad V = V_\tau, \&c. \qquad (47\,d)$$

But U_τ, V_τ, &c. are functions of ψ_τ, x_τ, y_τ, &c., from which, by the elimination of x_τ, y_τ, &c., the value of ψ_τ may be obtained in terms of U_τ, V_τ, &c.; and the required value of ψ is, consequently, the same function of U, V, &c.

2. Integrate (41 d) when the quantities T, X, Y, &c. are functions respectively of t, x, y, &c., each function involving but one variable.

Solution. In this case, (45 d) gives

$$\int_{\tau}^{t} \frac{1}{T} = \int_{x_\tau}^{x} \frac{1}{X} = \int_{y_\tau}^{y} \frac{1}{Y} = \&c., \tag{48 d}$$

whence the values of x, y, &c. are determined in terms of t and constants. The substitution of these values in M reduces it to a function of t, which may be denoted by M_t, and we have finally

$$\psi - \psi_\tau = \int_{\tau}^{t} \frac{M_t}{T}, \tag{49 d}$$

and the required value of ψ is obtained by substituting in (49 d) the values of x_τ, y_τ &c., obtained from (48 d).

The values of x_τ, y_τ, &c. may easily be derived from the values of x, y, &c. by changing τ, t, x_τ, y_τ, &c. into t, τ, x, y, &c.; for the values of x, y, &c. belong to one end of the interval $t-\tau$, in the same way in which x_τ, y_τ, &c. belong to the other end of the same interval, so that τ may be considered as the variable, while t is constant.

3. Integrate the equation

$$t\, D_x \psi - x\, D_t \psi = 0. \tag{50 d}$$

Ans. $\psi = f.\sqrt{(x^2 - t^2 + \tau^2)}$, (51 d)

in which $f.$ denotes the function which ψ is of x when t becomes τ.

4. Integrate the equation

$$x\, D_x \psi + t\, D_t x = 0.$$

Ans. $\psi = f.\left(\frac{\tau\, x}{t}\right)$,

where $f.$ has the same signification as in (51 d).

Equations of the first order.

5. Integrate the equation

$$t\,D_t\,\psi + x\,D_x\,\psi = n\,\psi.$$

$$Ans.\quad \psi = \left(\frac{t}{\tau}\right)^n f.\left(\frac{\tau\,x}{t}\right),$$

where $f.$ has the same signification as in (51 d).

6. Integrate the equation

$$a\,t\,D_t\,\psi + b\,x\,D_x\,\psi = n\,\psi. \qquad (52\,d)$$

$$Ans.\ \psi = \left(\frac{t}{\tau}\right)^{\frac{n}{a}} f.\left[x\left(\frac{\tau}{t}\right)^{\frac{b}{a}}\right], \qquad (53\,d)$$

where $f.$ has the same signification as in (51 d).

7. Integrate the equation

$$a\,t^h\,D_t\,\psi + b\,x^k\,D_x\,\psi = n\,\psi^m. \qquad (54\,d)$$

$$Ans.\ \psi = \left[\frac{1}{\psi_\tau^{m-1}} + \frac{n\,(m-1)}{a\,(h-1)}\left(\frac{1}{t^{h-1}} - \frac{1}{\tau^{h-1}}\right)\right]^{-(m-1)} \qquad (55\,d)$$

$$\text{where}\quad \psi_\tau = f.\left[\frac{1}{x^{k-1}} - \frac{b\,(k-1)}{a\,(h-1)}\left(\frac{1}{t^{h-1}} - \frac{1}{\tau^{h-1}}\right)\right]^{-(k-1)}$$

and $f.$ has the same signification as in (51 d).

8. Integrate the equation

$$t\,D_t\,\psi + \psi\,D_x\,\psi + x = 0.$$

$$Ans.\quad \psi^2 = x_\tau^2 - x^2 + (f.\,x_\tau)^2,$$

$$\text{where}\qquad x_\tau = x\,\cos.\log.\frac{t}{\tau} - \psi\,\sin.\log.\frac{t}{\tau},$$

and $f.$ has the same signification as in (51 d).

9. Integrate the equation

$$t\,(b + D_x\,\psi) - x\,(a + D_t\,\psi) + \psi\,(b\,D_t\,\psi - a\,D_x\,\psi) = 0. \qquad (56\,d)$$

Equations of the first order.

Solution. In this case (39 d and 40 d) give

$$\frac{1}{b\,\psi - x} = \frac{D_t\,x}{t - a\,\psi} = \frac{D_t\,\psi}{a\,x - b\,t}.$$

Hence $$D_t\,\psi + b\,D_t\,x + a = 0$$

and $$\psi\,D_t\,\psi + x\,D_t\,x + t = 0,$$

the integrals of which are

$$\psi - \psi_\tau + b\,(x - x_\tau) + a\,(t - \tau) = 0$$

$$\psi^2 - \psi_\tau^2 + x^2 - x_\tau^2 + t^2 - \tau^2 = 0.$$

Hence $$\psi_\tau + \psi = \frac{t^2 - \tau^2 + x^2 - x_\tau^2}{b\,(x - x_\tau) + a(t - \tau)};$$

and the required integral is obtained by the elimination of x_τ between the equations

$$[2\,\psi + b(x - x_\tau) + a(t - \tau)][b(x - x_\tau) + a(t - \tau)] = x^2 - x_\tau^2 + t^2 - \tau^2, \quad (57\,\mathrm{d})$$

$$\psi - f.\,x_\tau + b\,(x - x_\tau) + a(t - \tau) = 0, \quad (58\,\mathrm{d})$$

where $f.$ has the same signification as in (51 d).

Corollary. The integral of this equation is, by the corollary to the first example,

$$\psi + b\,x + a\,t = \varphi\,.\,(\psi^2 + x^2 + t^2), \quad (59\,\mathrm{d})$$

in which $\varphi\,.$ is an arbitrary function to be determined by the condition that

$$f.\,x + b\,x + a\,\tau = \varphi\,.\,[(f.\,x)^2 + x^2 + \tau^2], \quad (60\,\mathrm{d})$$

or it may be that $\psi + b\,x + a\,\tau$ is a given function φ of $\psi^2 + x^2 + \tau^2$.

10. Integrate the equation

$$a\,t\,D_t\,\psi + b\,x\,b_x\,\psi + c\,y\,D_y\,\psi + \&c. = n\,\psi. \quad (61\,\mathrm{d})$$

Equations of the first order.

Ans. $$\psi = \left(\frac{t}{\tau}\right)^{\frac{n}{a}} f.\left[\left(\frac{\tau}{t}\right)^{\frac{b}{a}} x, \left(\frac{\tau}{t}\right)^{\frac{c}{a}} y, \text{ \&c.}\right], \qquad (62\,\text{d})$$

in which $f.$ is the function which ψ is of x, y, &c. when t becomes τ.

Corollary. When

$$a = b = c = \text{\&c.} = 1, \qquad (63\,\text{d})$$

(62 d) becomes

$$\psi = \left(\frac{t}{\tau}\right)^n f.\left(\frac{\tau x}{t}, \frac{\tau y}{t}, \text{ \&c.}\right), \qquad (64\,\text{d})$$

so that ψ is, in this case, a homogeneous function of the nth degree, of t, x, y, &c., and (61 d) becomes, by the substitution of (63 d), a proposition applicable to such functions.

11. Integrate the equation

$$\frac{L+a\,l_t}{D_t\,l_t} D_t\,\psi + \frac{L+a\,l_x}{D_x\,l_x} D_x\,\psi + \frac{L+a\,l_y}{D_y\,l\,y} D_y\,\psi + \text{\&c.} = \pi, \qquad (65\,\text{d})$$

in which l_t is a given function of t, Π a given function of ψ, and L a given function of $l_t + l_x + l_y + \text{\&c.}$

Solution. Let $$M = l_t + l_x + l_y + \text{\&c.}, \qquad (66\,\text{d})$$

and (39 d and 40 d) give, in this case,

$$\frac{D_t\,l_t}{L+a\,l_t} = \frac{D_x\,l_x\,.\,D_t\,x}{L+a\,l_x} = \frac{D_y\,l_y\,.\,D_t\,y}{L+a\,l_y} = \text{\&c.} = \frac{D_t\,\psi}{\Pi}. \qquad (67\,\text{d})$$

Hence, by the theory of proportions, and since

$$D_t M = D_t\,l_t + D_x\,l_x\,.\,D_t\,x + D_y\,l_y\,.\,D_t\,y + \text{\&c.} \qquad (68\,\text{d})$$

$$\frac{D_t\,M}{n\,L+a\,M} = \frac{D_t\,.(l_t - l_x)}{a\,(l_t - l_x)} = \frac{D_t\,.(l_t - l_y)}{a\,(l_t - l_y)} = \text{\&c.} = \frac{D_t\,\psi}{\Pi}, \qquad (69\,\text{d})$$

where n is the number of the variables t, x, y, &c.

Equations of the first order.

Whence, since L is a function of M,

$$a\int_{M_\tau}^{M}\frac{1}{n\,L+a\,m}=\log.\frac{(l_t-l_x)}{l_\tau-l_{x_\tau}}=\log.\frac{l_t-l_y}{l_\tau-l_{y_\tau}}=\&c.=a\int_{\psi_\tau}^{\psi}\frac{1}{\Pi}, \tag{70 d}$$

where $$\psi_\tau=f.(x_\tau,\,y_\tau,\,\&c.), \tag{71 d}$$

f having the same signification as in (52 d); and the required integral is obtained by the elimination of x_τ, y_τ, &c. between the equations (70 d and 71 d).

12. Integrate the equation (65 d) when

$$L=m\,M,\text{ and }\Pi=b\,\psi. \tag{72 d}$$

Ans. The equation obtained by eliminating M_τ between

$$\psi=\left(\frac{M}{M_\tau}\right)^{\frac{b}{m\,n+a}}f'.\left[l_t-\left(\frac{M_\tau}{M}\right)^{\frac{a}{m\,n+a}}(l_t-l_x),\,l_\tau-\left(\frac{M_\tau}{M}\right)^{\frac{a}{m\,n+a}}(l_t-l_y)\ \&c.\right] \tag{73 d}$$

$$\frac{n\,l_t-M}{n\,l_\tau-M_\tau}=\left(\frac{M}{M_\tau}\right)^{\frac{a}{m\,n+a}}, \tag{74 d}$$

where f' is the function which ψ becomes of l_x, l_y, &c., when t becomes τ.

When $$m\,n+a=0,$$

the integral becomes

$$\psi=\left(\frac{n\,l_t-M}{n\,l_\tau-M}\right)^{\frac{b}{a}}f.\left[l_x+\left(\frac{M(l_t-l_\tau}{n\,l_t-M}\right),\,l\,y+\frac{M\,(l_t-l_\tau)}{n\,l_\tau-M},\,\&c.\right] \tag{75 d}$$

13. Integrate the equation (65 d), when L is any given function of the variables t, x, y, &c., and ψ.

Solution. In this case, the first member of (69 d) must be omitted, but the other members give the values of all the va-

riables expressed in terms of any two of them, and, therefore, the value of L expressed in terms of these two, which two may, for instance, be ψ and t, and the equation

$$\frac{D_t\, l_t}{L+a\, l_t} = \frac{D_t\, \psi}{\pi} \tag{76 d}$$

will be a differential equation of the first order between two variables, and may admit of easy integration.

Corollary. This method may be applied in any case, in which *all the integrals but one of a system of differential equations of the first order have been obtained,* and *the final integral will depend only upon the integration of a differential equation of the first order between two variables.*

14. Integrate the equation (65 d), when L is a given function of $l_t + l_x + l\,y + \&c. + l_\psi$, and

$$\Pi = \frac{L + a\, l_\psi}{D_\psi\, l_\psi}. \tag{77 d}$$

Ans. The equation obtained by eliminating x_τ, y_τ, &c. between the equations

$$e^{aN} = \frac{l_t - l_x}{l_\tau - l_{x_\tau}} = \frac{l_t - l_y}{l_\tau - l_{y_\tau}} = \&c. = \frac{l_t - l_\psi}{l_\tau - l_{\psi_\tau}}, \tag{78 d}$$

where ψ_τ has the same signification as in (71 d), and

$$N = \int_{M_\tau}^{M} \frac{1}{n\, L + a\, M} \tag{79 d}$$

$$M = l_t + l_x + l_y + \&c. + l_\psi, \tag{80 d}$$

and n is the number of the variables t, x, y, &c., and ψ.

15. Integrate the preceding example when

$$l_t = m\,t, \quad M = L. \tag{81 d}$$

Ans. The equation, obtained by eliminating L_τ between

$$\left(\frac{L}{L_\tau}\right)^{\frac{a}{nm+a}} = \frac{t-\psi}{\tau-\psi_\tau} = \frac{n\,t-L}{n\,\tau-L_\tau}, \tag{82 d}$$

where ψ_τ is the same as in (71 d).

When

$$m\,n + a = 0, \tag{83 d}$$

the integral is

$$\frac{t-\psi}{\tau-\psi_\tau} = \frac{n\,t-L}{n\tau-L}. \tag{84 d}$$

16. Integrate the equation

$$T + X + Y + \&c. = 0, \tag{85 d}$$

in which T is a function of t and $D_t\,\psi$, X a function of x and $D_x\,\psi$, Y a function of y and $D_y\,\psi$, &c.

Solution. In this case (39 d and 40 d) become

$$\frac{1}{D_s\,T} = \frac{D_t\,x}{D_p\,X} = \frac{d_t\,y}{D_q\,Y} = \&c. \tag{86 d}$$

$$= -\frac{D_t\,p}{D_x X} = -\frac{D_t\,q}{D_y\,Y} = \&c. = \frac{D_t\,\psi}{s\,D_s\,T + p\,D_p\,X + q\,D_q\,Y + \&c.}$$

which give the equations

$$D_x\,X . D_t\,x + D_p\,X . D_t\,p = 0$$

$$D_y\,Y . D_t\,y + D_q\,Y . D_t\,q = 0, \;\&c., \tag{87 d}$$

the integrals of which are, by denoting the values which X, Y, &c. have when x, y, &c., p, q, &c. become x_τ, y_τ, &c. by X_τ, Y_τ, &c.,

$$X = X_\tau, \; Y = Y_\tau, \; \&c. \tag{88 d}$$

Equations of the first order.

Hence, by (85 d),

$$T = T_\tau. \tag{89 d}$$

These equations give s, p, q, &c. in terms respectively of t, x, y, &c., which substituted in (86 d) give, by integration,

$$\int_\tau^t \cdot \frac{1}{D_s T} = \int_{x_\tau}^x \frac{1}{D_p X} = \int_{y_\tau}^y \frac{1}{D_q Y} = \&c. \tag{90 d}$$

These equations give x, y, &c. in terms of t, which, substituted in (86 d) give

$$\psi - \psi_\tau = \int_\tau^t \frac{s\, D_s T + p\, D_p X + q\, D_q Y + \&c.}{D_s T}, \tag{91 d}$$

where ψ_τ is the same as in (71 d). The integral of the given equation is, finally, the equation obtained by eliminating x_τ, y_τ, &c. between (90 d and 91 d). In making this elimination, it is to be observed that

$$p_\tau = D_{x_\tau} \psi_\tau, \quad q_\tau = D_{y_\tau} \cdot \psi_\tau, \&c. \tag{92 d}$$

17. Integrate (85 d), when T is a function of $D_t \psi$, X of $D_x \psi$, &c.

Solution. In this case (86 d) gives

$$p = p_\tau, \quad q = q_\tau, \&c. \tag{93 d}$$

$$x - x_\tau = \frac{D_p X}{D_s T} (t - \tau),$$

$$y - y_\tau = \frac{D_q Y}{D_s T} (t - \tau), \&c., \tag{94 d}$$

$$\psi - \psi_\tau = \frac{s\, D_s T + p\, D_p X + q\, D_q Y + \&c.}{D_s T} (t - \tau). \tag{95 d}$$

Hence, if ψ_τ is used as in (71 d), and if

$$f'(x_\tau, y_\tau, \&c.) = D_{x_\tau} f(x_\tau, y_\tau, \&c.)$$

$$f''(x_\tau, y_\tau, \&c.) = D_{y_\tau} f(x_\tau, x_\tau, \&c.)\ \&c., \quad (96\text{ d})$$

the required integral is obtained by eliminating $s, p, q,$ &c. between the given equation and the equations

$$\psi - f.\left[x - \frac{D_p X}{D_s T}(t-\tau),\ y - \frac{D_q Y}{D_s \tau}(t-\tau),\ \&c.\right]$$
$$= \frac{s\,D_s T + p\,D_p X + q\,D_q Y}{D_s T}(t-\tau), \quad (97\text{ d})$$

$$p = f.'\left[x - \frac{D_p X}{D_s T}(t-\tau),\ y - \frac{D_q Y}{D_s T}(t-\tau),\ \&c.\right]$$

$$q = f.''\left[x - \frac{D_p X}{D_s T}(t-\tau),\ y - \frac{D_q Y}{D_s T}(t-\tau),\ \&c.\right]\&c. \quad (98\text{ d})$$

18. Integrate the equation

$$D_t \psi = a\,(D_x \psi)^2.$$

Ans. The equation obtained by eliminating p between the equations

$$\psi = f.\,[x + 2\,a\,p\,(t-\tau)] - a\,p^2\,(t-\tau)$$

and $$p = f.'\,[x + 2\,a\,p\,(t-\tau)],$$

where f and f' are used as in (71 d and 95 d).

19. Integrate the equation

$$(D_t \psi)^2 = b\,(D_x \psi)^2. \quad (99\text{ d})$$

$$\textit{Ans.}\ \psi = f.\,[x + (t-\tau)\sqrt{b}], \quad (1\text{ e})$$

where f is used as in (51 d).

20. Integrate the equation

$$R = 0, \tag{2 e}$$

where R is a function of T, X, Y, &c., which have the same signification as in (85 d).

Solution. It may easily be shown that the equations (87 d, 88 d and 89 d) are applicable to this case. Hence the values of s, p, q, &c. may be found in terms respectively of t, x, y, &c., and these values, substituted in (32 d), reduce the successive terms to functions respectively of t, x, y, &c. The integral of (32 d) gives, therefore,

$$\psi - \psi_\tau = \int_\tau^t \cdot s + \int_{x_\tau}^x p + \int_{y_\tau}^y q + \&\text{c.}, \tag{3 e}$$

and the value of ψ, obtained by eliminating p_τ, q_τ, &c. between (2 e, 3 e, and 92 d), satisfies (2 e). The values of x_τ, y_τ, &c. are finally eliminated by means of the integrals of the upper line of (39 d), which is freed from s, p, q, &c. by means of (88 d and 89 d).*

The functions $D_T R$, $D_X R$, &c., are functions of T, X, &c., and therefore by (88 d and 89 d) they are constant, so that the integrals of the upper line of (39 d) become

$$\frac{1}{D_T R}\int_\tau^t \cdot \frac{1}{D_s T} = \frac{1}{D_X R}\int_{x_\tau}^x \cdot \frac{1}{D_p X} = \frac{1}{D_Y R}\int_{y_\tau}^y \cdot \frac{1}{D_q X} = \&\text{c.} \tag{4 e}$$

and the required integral is therefore the result of the elimination of x_τ, y_τ, &c. between the equations obtained from (3 e

* *Note.* This last process, which is necessary in order that ψ may become a given function of x, y, &c. when t becomes τ, is neglected in the ordinary solution of this question given in (Lacroix, Calc. Diff. et Int., 2d ed., Vol. I, p. 572).

and 4 e) by the substitution of (92 d) and of the value of s_τ obtained from (2 e) by changing T, X, Y, &c. to T_τ, X_τ, Y_τ, &c.

21. Integrate the preceding example when T, X, Y, &c. are the same as in example 17.

Ans. The integral in the equation obtained by the elimination of x_τ, y_τ, &c. and s_τ, between the equations obtained from

$$\psi = \psi_\tau + s_\tau (t-\tau) + p_\tau (x-x_\tau) + q_\tau (y-y_\tau) + \&c. \quad (5\,e)$$

$$\frac{t-\tau}{D_{s_\tau} R_\tau} = \frac{x-x_\tau}{D_{p_\tau} R_\tau} = \frac{y-y_\tau}{D_{q_\tau} R_\tau} = \&c. \quad (6\,e)$$

and
$$R_\tau = 0, \quad (7\,e)$$

by the substitution of (92 d).

22. Integrate example 20, when

$$T = T' D_t \psi, \quad X = X' D_x \psi, \&c. \quad (8\,e)$$

where T', X', Y', &c. are functions, respectively, of t, x, y, &c.

Ans. The integral is the equation obtained by the elimination of s_τ, x_τ, y_τ, &c. between the equations obtained from

$$\psi = \psi_\tau + \left(s_\tau T'_\tau + \frac{p_\tau X'_\tau D_{x_\tau} R_\tau + q_\tau Y'_\tau D_{y_\tau} R_\tau + \&c.}{D_{T_\tau} R_\tau}\right) \int_\tau^t \frac{1}{T'} \quad (9\,e)$$

$$\frac{1}{D_{T_\tau} R_\tau} \int_\tau^t \frac{1}{T'} = \frac{1}{D_{x_\tau} R_\tau} \int_{x_\tau}^x \frac{1}{X'} = \&c. \quad (10\,e)$$

and
$$R_\tau = 0, \quad (11\,e)$$

by the substitution of (92 d).

Equations of the first order.

23. Integrate example 20, when

$$T = T'(D_t\psi)^n, \quad X = X'(D_x\psi)^m \text{ \&c.,} \qquad (12\,e)$$

where T', X', &c. are the same as in the preceding example.

Ans. The integral is the equation obtained by the elimination of s_τ, x_τ, y_τ, &c. between the equations obtained from

$$\psi = \psi_\tau + \left(s_\tau \sqrt[n]{T'_\tau} + \frac{m\,p_\tau^m X'_\tau D_{x_\tau} R_\tau + \&c.}{n\,s_\tau^{n-1} \sqrt[n]{T'_\tau{}^{(n-1)}}}\right) \int_\tau^t \frac{1}{\sqrt[n]{T'}} \qquad (13\,e)$$

$$\frac{1}{n\,s_\tau^{n-1}\,T'_\tau{}^{\frac{n-1}{n}}\,D_{T_\tau} R_\tau} \int_\tau^t \frac{1}{\sqrt[n]{T'}}$$

$$= \frac{1}{m\,p_\tau^{m-1}\,X'_\tau{}^{\frac{m-1}{m}}\,D_{x_\tau} R_\tau} \int_{x_\tau}^x \frac{1}{\sqrt[m]{X'}} = \&c. \qquad (14\,e)$$

and (11 e) by the substitution of (92 d).

24. To integrate the equation

$$R = 0, \qquad (15\,e)$$

when R is a function of t, x, y, &c., D_t, ψ, $D_x\psi$, &c. and φ where

$$\varphi = t\,D_t\psi + x\,D_x\psi + \&c. - \psi. \qquad (16\,e)$$

Solution. If s, p, q, &c. have the same signification as in § 222, (16 e) gives

$$\varphi = t\,s + x\,p + \&c. - \psi, \qquad (17\,e)$$

and if the differentials are taken, as if s, p, q, &c. were the independent variables of which t, x, &c. are functions, we have

$$D_s\varphi = t + s\,D_s t + p\,D_s x + \&c. - D_t\psi\,D_s t - D_x\psi\,D_s x - \&c.$$
$$= t, \qquad (18\,e)$$

and, in the same way,

$$D_p\,\varphi = x, \quad D_q\,\varphi = y, \text{ \&c.}; \qquad (19\,\text{e})$$

that is, t, x, y, &c. are the differential coefficients of φ relatively to s, p, q, &c., and, therefore, *the integral of* (15 e) *may be obtained as if* φ *were the unknown function*, s, p, q, *&c. the independent variables, and* t, x, y, *&c. the respective differential coefficients.*

224. When the required function ψ is dependent upon several variables, there may be several given equations between its differential coefficients, and the solution is possible, provided the number of equations does not exceed the number of variables. In this case of several *simultaneous equations*, as many differential coefficients may be eliminated as the number of equations exceed unity; and the resulting equation may be integrated by the preceding methods. It is to be observed that, in the integration of this equation, those variables may be regarded as constant, of which the corresponding differential coefficients have been eliminated. The relation of the required function to the variables, which have been thus regarded as constant, is determined by substitution in the given equations of the result of the integration. The limits of this volume do not, however, permit any examples of this process.

CHAPTER XII.

INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

225. *Problem. To find a function ψ of two variables, x and t, which satisfies a given differential equation of the second order, and which becomes a given function of x for a given function τ of t, and its first differential coefficient $D_t \psi$, taken relatively to t, becomes another given function of x for the same value of t.*

Solution. Let

$$\left.\begin{array}{l} s = D_t \psi, \quad p = D_x \psi, \\ \sigma = D_t^2 \psi, \quad \varrho = D_{t,\,x}^2 \psi, \quad \xi = D_x \psi, \end{array}\right\} \qquad (20\text{ e})$$

and let the given equation, by the substitution of these values, become

$$R = 0. \qquad (21\text{ e})$$

If D' denotes the differential coefficient of a function taken relatively to either of the given functions of x, we have

$$D' \psi = p\, D' x, \quad D' p = \xi\, D' x, \quad D' s = \varrho\,.\,D' x. \qquad (22\text{ e})$$

Although x is independent of t, it may be assumed to be an arbitrary function of t, and in this hypothesis ψ, p, s, &c. will become functions of t, and will give

$$D_t \psi = s + p\, D_t x, \qquad (23\text{ e})$$

$$\left.\begin{array}{l} D_t \psi = \varrho + \xi\, D_t x \\ D_t s = \sigma + \varrho\, D_t x. \end{array}\right\} \qquad (24\text{ e})$$

Equations of the second order.

The differential coefficients of (24 e) are

$$\left.\begin{aligned} D' D_t p &= D' \varrho + D' \xi \,.\, D_t x + \xi D' D_t x \\ D' D_t s &= D' \sigma + D' \varrho \,.\, D_t x + \varrho D' D_t x, \end{aligned}\right\} \quad (25\,\mathrm{e})$$

and those of (22 e) give

$$\left.\begin{aligned} D' D_t p &= D_t \xi\, D' x + \xi D' D_t x \\ D' D_t s &= D_t \varrho\, D' x + \varrho D' D_t x\,; \end{aligned}\right\} \quad (26\,\mathrm{e})$$

which, substituted in (25 e), give

$$D' \varrho = D_t \xi \,.\, D' x - D_t x \,.\, D' \xi\,, \quad (27\,\mathrm{e})$$

$$\begin{aligned} D' \sigma &= D_t \varrho \,.\, D_t x - D_t x \,.\, D' \varrho \\ &= (D_t \varrho - D_t \xi \,.\, D_t x)\, D' x + (D_t x)^2 D' \xi. \quad (28\,\mathrm{e}) \end{aligned}$$

If the values of $D' \psi$, $D' p$, $D' s$, $D' \varrho$ and $D' \sigma$ (22 e, 27 e, 28 e) are substituted in

$$D' R = 0, \quad (29\,\mathrm{e})$$

the resulting equation contains the two arbitrary and independent elements $D' x$ and $D' \xi$, the coefficients of which, being put equal to zero, give the two equations

$$D_\sigma R\, (D_t x)^2 - D_\varrho R \,.\, D_t x + D_\xi R = 0, \quad (30\,\mathrm{e})$$

$$\begin{aligned} D_\sigma R\, (D_t \varrho - D_t \xi \,.\, D_t x) + D_\varrho R\, D_t \xi + \varrho D_s R + \xi D_p R \\ + p\, D_\psi R + D_x R = 0. \quad (31\,\mathrm{e}) \end{aligned}$$

Whenever, from a judicious combination of the equations (21 e, 23 e, 24 e, 30 e and 31 e), three equations can be found capable of integration, the elimination of ϱ, σ and ξ between the three integrals of these equations and the equation (21 e) will give two equations between p, s, ψ, x, x_τ, ψ_τ, s_τ, p_τ, ϱ_τ and ξ_τ. In the two equations thus obtained, $D_t \psi$ and $D_x \psi$ may be substituted for s and p, $D_{x_\tau} \psi_\tau$ for p_τ, $D^2_{x_\tau} \psi_\tau$ for ξ_τ,

$D_{x_\tau} s_\tau$ for ϱ_τ, and two functions of x_τ for ψ_τ and s_τ corre ponding to the given functions of x, which ψ and s becon when t becomes τ. Between the two equations thus obtaine x_τ may be eliminated, and the resulting equation is a differe tial equation of the first order, and its integral, obtained b the methods of the preceding chapter, is the required int gral.

This process is precisely similar to that of § 222, and i derived from the same principles.

226. *Corollary.* The two given functions of x are wholl arbitrary, and may be altogether independent of each other They involve, therefore, in the general value of ψ, two inde pendent and arbitrary functions of ψ, x and t, and which ma be independent, not merely in reference to the nature of the functional operations themselves, but in regard to the variables, that is, to the combinations of ψ, x and t, upon which they depend. This variable or combination is represented by x_τ in the preceding section. There must, therefore, in general, be two different values of x_τ, each of which will give a different equation of the first order, the integral of either of which leads to the required value of ψ. But instead of integrating the two equations of this first order independently of each other, it will be found much easier to integrate either of the equations obtained from them by the elimination of $D_x \psi$ or $D_t \psi$.

227. *Scholium.* There are many cases in which it is expedient to transform the given equation, before applying this process of integration; and some of them will be considered among the examples.

Whenever the three required integrals cannot be obtained, the preceding process is inapplicable, although the given equation may sometimes admit of integration in these cases, by means of analytical artifices.

228. Examples.

1. Integrate the equation

$$D_t^2\,\psi + (a+b)\,D_{t,x}^2\,\psi + a\,b\,D_x^2\,\psi = P, \qquad (32\text{ e})$$

in which P is a given function of x and t.

Solution. In this case the equation (21 e) is

$$\sigma + (a+b)\,\varrho + a\,b\,\xi = P, \qquad (33\text{ e})$$

and (29 e) is

$$D'\sigma + (a+b)\,D'\varrho + a\,b\,D'\xi = D_x\,P\,.\,D'\,x. \qquad (34\text{ e})$$

Hence, by the substitution of (27 e and 28 e), the coefficient of $D'\xi$ placed equal to zero, is

$$(D_t\,x)^2 - (a+b)\,D_t\,x + a\,b = 0\,; \qquad (35\text{ e})$$

whence

$$D_t\,x = a \text{ or } = b, \qquad (36\text{ e})$$

$$x - x_\tau = a\,(t-\tau) \text{ or } = b\,(t-\tau). \qquad (37\text{ e})$$

The first of these two values reduces (23 e and 24 e) to

$$\left.\begin{array}{l} D_t\,p = \varrho + a\,\xi \\ D_t\,s = \sigma + a\,\varrho = -b\,\varrho - a\,b\,\xi + P = -b\,D_t p + P, \end{array}\right\} \qquad (38\text{ e})$$

$$D_t\,s + b\,D_t\,p = P, \qquad (39\text{ e})$$

in which P may by (37 e) be reduced to a function of t, and denoted by P_t; hence

$$s - s_\tau + b\,(p - p_\tau) = \textstyle\int_\tau^t P_t\,. \qquad (40\text{ e})$$

In the same way, if s'_τ, p'_τ and P'_t denote the corresponding values when the second equations in (36 e and 37 e) are employed, we find

$$s - s'_\tau + a\,(p - p'_\tau) = \textstyle\int_\tau^t P'_t\,. \qquad (41\text{ e})$$

These two equations become, by the substitution of $f.x$ for the values of ψ and $D_t\psi$ when t is τ,

$$D_t\psi+b\,D_x\psi-f_0(x-a\,t+a\,\tau)-b\,D_x f(x-a\,t+a\,\tau)=Q, \quad (42\,e)$$

$$D_t\psi+a\,D_x\psi-f_0(x-b\,t+b\,\tau)-a\,D_x f(x-b\,t+b\,\tau)=Q', \quad (43\,e)$$

in which Q and Q' are the values which the second members of (40 e and 41 e) acquire by the substitution for x_τ of its value given by (37 e). The combination of these two equations gives

$$(a-b)D_t\psi-a\,f_0(x-a\,t+a\,\tau)+b\,f_0(x-b\,t+b\,\tau)$$
$$-a\,b\,D_x f.(x-a\,t+a\,\tau)+a\,b\,D_x f(x-b\,t+b\,\tau)=a\,Q-b\,Q', \quad (44\,e)$$

$$(a-b)\,D_x\psi+f_0(x-a\,t+a\,\tau)-f_0(x-b\,t+b\,\tau)$$
$$+b\,D_x f(x-a\,t+a\,\tau)-a\,f(x-b\,t+b\,\tau)=Q'-Q, \quad (45\,e)$$

the integral of which is

$$(a-b)\,\psi+\textstyle\int_x f_0(x-a\,t+a\,\tau)-\int_x f_0(x-b\,t+b\,\tau)$$
$$+b\,f(x-a\,t+a\,\tau)-a\,f.(x-a\,t+a\,\tau)=\textstyle\int_\tau^t(a\,Q-b\,Q'). \quad (46\,e)$$

2. Integrate the preceding example when

$$P=t\,x.$$

Ans. The equation (46 e) when its second member becomes

$$\tfrac{1}{2}(a-b)\,(t-\tau)^2\,[\tfrac{1}{3}\,x\,(t+2\,\tau)-\tfrac{1}{12}\,(a+b)\,(t+3\,\tau)\,(t-\tau)].$$

3. Integrate the equation

$$t^2\,D_t^2\psi+2\,t\,x\,D_{t.x}^2\psi+x^2\,D_x^2\psi=P,$$

in which P is a given function of x and t.

Equations of the second order.

$$Ans.\ \psi = f.\frac{x\,\tau}{t} + \left(f_0.\frac{x\,\tau}{t} + \frac{x}{t}\,D_x f.\frac{x\,\tau}{t}\right)(t-\tau) + Q,$$

in which P_t is the value of P when x is changed to $\frac{t\,x_\tau}{\tau}$, Q is the value of

$$\int_\tau^t \int_\tau^t \frac{P_t}{t^2},$$

when $\frac{x\,\tau}{t}$ is substituted for x_τ.

4. Integrate the equation

$$(D_x\psi)^2.D_t^2\psi - 2\,D_x\psi.D_t\psi.D_{t.x}^2\psi + (D_t\psi)^2\,D_x^2\,\psi = P(D_x\psi)^3, \tag{47 e}$$

in which P is a function of $\frac{D_t\psi}{D_x\psi}$.

Solution. In this case (30 e) becomes by (20 e),

$$p^2\,(D_t\,x)^2 + 2\,p\,s\,D_t\,x + s^2 = 0,$$

whence

$$D_t\,x = -\frac{s}{p}, \tag{48 e}$$

and by (23 e, 24 e and 27 e),

$$D_t\,\psi = 0, \tag{49 e}$$

$$D_t\,p = \varrho - \frac{s\,\xi}{p}, \tag{50 e}$$

$$D_t\,s = \sigma - \frac{s\,\varrho}{p} = \frac{\varrho\,s}{p} - \frac{s^2\,\xi}{p^2} + P\,p = \frac{s}{p}\,D_t p + P\,p. \tag{51 e}$$

Hence

$$D_t.\frac{s}{p} = P \qquad \text{and} \qquad \frac{1}{P}\,D_t.\frac{s}{p} = 1; \tag{52 e}$$

and since P is a function of $\frac{s}{p}$, the integral of (52 e) may be directly obtained, and gives $\frac{s}{p}$ in terms of t, and this value substituted in (48 e) gives

$$x - x_\tau = -\int_\tau^t \frac{s}{p}, \tag{53 e}$$

whence, by (49 e),

$$\psi = \psi_\tau = f \,.\, x_\tau; \tag{54 e}$$

and the required integral is obtained by the elimination of x_τ between (53 e and 54 e).

5. Integrate (47 e) when the second member is zero.

$$\textit{Ans.} \quad x = F.\psi - \frac{f'\,.\,F.\psi}{f_0\,.\,F.\psi}(t - \tau), \tag{55 e}$$

in which F is the inverse function of f, and f' is the differential coefficient of F taken with respect to its variable.

6. Integrate the equation

$$D_t^2\,\psi\,.\,D_x^2\,\psi - (D_{t.x}^2\,\psi)^2 = 0. \tag{56 e}$$

Solution. In this case (21 e) is

$$\sigma\,\xi - \varrho^2 = 0, \tag{37 e}$$

and (30 e) is

$$\xi\,(D_t\,x)^2 + 2\,\varrho\,D_t\,x + \sigma = 0,$$

whence $\qquad \varrho^2\,(D_t\,x)^2 + 2\,\varrho\,\sigma\,D_t\,x + \sigma^2 = 0$

$$D_t\,x = -\frac{\sigma}{\varrho},$$

and (23 e and 24 e) give

$$D_t\,s = 0, \quad s = s_\tau$$

$$D_t\,p = 0, \quad p = p_\tau$$

$$D_t\psi = s_\tau + p_\tau\,D_t\,x$$

$$\psi - \psi_\tau = s_\tau\,(t - \tau) + p_\tau\,(x - x_\tau),$$

and (31 e) gives

$$\frac{\sigma}{\varrho} = \frac{\sigma_\tau}{\varrho_\tau},$$

whence

$$x - x_\tau = \frac{\sigma_\tau}{\varrho_\tau}(t-\tau) = \frac{\varrho_\tau}{s_\tau}(t-\tau);$$

and the required integral is the result of the elimination of x_τ between the two equations

$$x - x_\tau = \frac{f_0' \,.\, x_\tau}{f'' \,.\, x_\tau}(t-\tau), \tag{58 e}$$

$$\psi - f . x_\tau = f_0 . x_\tau (t-\tau) + f' . x_\tau (x - x_\tau), \tag{59 e}$$

in which the accents denote the successive differential coefficients of the functions.

7. Integrate the equation

$$\psi D_t^2 \psi - \psi D_x^2 \psi = (D_t \psi)^2 - (D_x \psi)^2,$$

when the value of ψ becomes $f x . f_0 x$ when t becomes τ, and the value of $D_t \psi$ becomes

$$f' x . f_0 x - f x . f_0' x,$$

in which f' and f_0' are the differential coefficients of f and f_0.

Ans. $\psi = f(x+t-\tau) . f_0(x-t+\tau)$.

8. Integrate the equation

$$D_t^2 \psi - D_x^2 \psi - \frac{2}{t} D_t \psi = 0. \tag{60 e}$$

Solution. In this case, the general form of solution is inapplicable without previous transformation. For this purpose, by putting

$$\psi' = D_t \psi, \tag{61 e}$$

the differential coefficient of (60 e) relatively to t is

$$D_t^2\,\psi' - D_x^2\,\psi' - \frac{2}{t}\,D_t\,\psi' + \frac{2}{t^2}\,\psi' = 0. \qquad (62\text{ e})$$

The integral of (62 e) may be found by the general process, which gives

$$D_t\,x = \pm 1, \quad x - x_\tau = \pm(t - \tau), \qquad (63\text{ e})$$

$$D_t\,s = \sigma \pm \varrho = \pm D_t\,p + \frac{2\,s}{t} - \frac{2\,\psi'}{t^2}, \qquad (64\text{ e})$$

$$D_t\,p = \varrho \pm \xi, \qquad (65\text{ e})$$

$$D_t\,\psi' = s \pm p, \qquad (66\text{ e})$$

and the remainder, after subtracting (66 e), divided by t^2 from (64 e) divided by t, is

$$\frac{t\,D_t s - s}{t^2} = \pm\left(\frac{t\,D_t p - p}{t^2}\right) + \frac{t\,D_t\,\psi' - 2\,\psi'}{t^3}, \qquad (67\text{ e})$$

the integral of which is

$$\frac{s}{t} - \frac{s_\tau}{\tau} = \pm\left(\frac{p}{t} - \frac{p_\tau}{\tau}\right) + \frac{\psi'}{t^2} - \frac{\psi'_\tau}{\tau^2}. \qquad (68\text{ e})$$

The sum of the equations involved in (68 e) is

$$\frac{2\,D_t\,\psi'}{t} - \frac{f_2(x-t+\tau) + f_2(x+t-\tau)}{\tau} = \frac{f_1'(x+t-\tau) - f_1'(x-t+\tau)}{\tau}$$

$$+ \frac{2\,\psi'}{t^2} - \frac{f_1(x-t+\tau) + f_1(x+t-\tau)}{\tau^2}, \qquad (69\text{ e})$$

in which $f_1\,x$, and $f_2\,x$ denote the values of ψ' and $D_t\,\psi'$ when t is τ, and $f_2'\,x$ is the differential coefficient of $f_2\,x$ relatively to x.

Equations of the second order.

The integral of (69 e) is

$$\frac{2\tau\psi'}{t} = F_2(x+t-\tau) - F_2(x-t+\tau) + f_1(x+t-\tau) + f_1(x-t+\tau) \frac{-F_1(x+t-\tau) - F_1(x-t+\tau)}{\tau}; \quad (69\,e')$$

in which $F_2 x$ and $F_1 x$ are the integrals of $f_2 x$ and $f_1 x$ relatively to x. But it follows from (60 e, 61 e and 63 e) that

$$f_0 x = f_1 x \quad (70\,e)$$

$$f_2 x = f''_{\cdot} x + \frac{2}{t} f_0 x, \quad (71\,e)$$

whence (69 e′) becomes

$$\frac{2\tau D_t \psi}{t} = F''(x+t-\tau) - F_2(x-t+\tau) + f_0(x+t-\tau) + f_0(x-t+\tau) + \frac{F_0(x+t-\tau) - F_0(x-t+\tau)}{\tau}, \quad (72\,e)$$

and ψ is obtained from the integral of (72 e).

9. Integrate the equation

$$P D_t^2 \psi + S D_{t.x}^2 \psi + T D_x^2 \psi = 0, \quad (73\,e)$$

in which P, S, and T are functions of $D_t \psi$, and $D_x \psi$.

Solution. Let

$$\varphi = s t + p x - \psi, \quad (74\,e)$$

in which s and p have the same signification as in (20 e).

And if the differentials are taken as if s and p are the independent variables, of which t and x are functions, we have, as in (18 e and 19 e),

$$D_s \varphi = t, \quad D_p \varphi = x. \quad (75\,e)$$

The differentials of (75 e), taken as if t and x are the independent variables, while the first members are expressed in terms of p and s, are

$$\left.\begin{aligned} 1 &= D_t\, D_s\, \varphi = D_s^2\, \varphi\, D_t\, s + D_{s.p}^2\, \varphi\, D_t\, p \\ 0 &= D_t\, D_p\, \varphi = D_{s.p}^2\, \varphi\, D_t\, s + D_p^2\, \varphi\, D_t\, p \end{aligned}\right\} \quad (76\text{ e})$$

$$\left.\begin{aligned} 0 &= D_x\, D_s\, \varphi = D_s^2\, \varphi\, D_x\, s + D_{s.p}^2\, \varphi\, D_x\, p \\ 1 &= D_x\, D_p\, \varphi = D_{s.p}^2\, \varphi\, D_x\, s + D_p^2\, \varphi\, D_x\, p, \end{aligned}\right\} \quad (77\text{ e})$$

and (76 e) and (77 e) give

$$\left.\begin{aligned} D_t^2\, \psi &= D_t\, s = M\, D_p^2\, \varphi \\ D_{t.x}^2\, \psi &= D_t\, p = D_x\, s = -M\, D_{p.s}^2\, \varphi \\ D_x^2\, \psi &= D_x\, p = M\, D_s^2\, \varphi, \end{aligned}\right\} \quad (78\text{ e})$$

in which

$$\frac{1}{M} = (D_{s.p}^2\, \varphi)^2 - D_s^2\, \varphi\, .\, D_p^2\, \varphi, \qquad (79\text{ e})$$

and the substitution of (78 e) in (74 e) gives

$$P\, D_p^2\, \varphi - S\, D_{p.s}^2\, \varphi + T\, D_s^2\, \varphi = 0, \qquad (80\text{ e})$$

which may be integrated as if p and s were the independent variables.

10. *Integrate the equation of the surface of minimum extent.*

Solution. By changing, in (959), x, y and z to t, x and ψ, to correspond to the notation of this section, that equation becomes

$$(1+(D_x\, \psi)^2)\, D_t^2\, \psi - 2\, D_t\, \psi . D_x\, \psi\, D_{t.x}^2\, \psi + (1+(D_t\, \psi)^2) D_x^2\, \psi = 0. \qquad (81\text{ e})$$

By the substitution of the preceding example this equation becomes

$$(1+p^2)\, D_p^2\, \varphi + 2\, p\, s\, D_{p.s}^2\, \varphi + (1+s^2)\, D_s^2\, \varphi = 0, \quad (82\text{ e})$$

Integration of equation of surface of minimum extent.

which cannot be integrated by the direct application of the general process. If, however, we put

$$\varphi' = D_s\, \varphi, \tag{83 e}$$

the differential coefficient of (82 e), relatively to s, is

$$(1+p^2)\, D_p^2\, \varphi' + 2\, p\, s\, D_{p.s}^2\, \varphi' + (1+s^2)\, D_s^2\, \varphi' + 2\, p\, D_p\, \varphi' + 2\, s\, D_s\, \varphi' = 0. \tag{84 e}$$

The general process applied to this case gives, by putting

$$\left.\begin{array}{lll} s' = D_s\, \varphi', & p' = D_p\, \varphi' & \\ \sigma = D_s^2\, \varphi', & \varrho = D_{p.s}^2\, \varphi', & \xi = D_p^2\, \varphi', \end{array}\right\} \tag{85 e}$$

$$(1+s^2)\, (D_s p)^2 - 2\, p\, s\, D_s p + (1+p^2) = 0, \tag{86 e}$$

the integral of which, found by the process of Ex. 27 of § 220, is

$$p = \frac{p - p_\tau}{s - s_\tau}\, s \pm \sqrt{\left[-1 - \left(\frac{p - p_\tau}{s - s_\tau}\right)^2\right]}, \tag{87 e}$$

in which s_τ and p_τ should be accented when the lower sign is used.

Instead of proceeding with the direct process, we may put

$$m = \frac{p - p_\tau}{s - s_\tau}, \quad n = \frac{p - p'_\tau}{s - s'_\tau}, \tag{88 e}$$

and (87 e) gives

$$p = m\, s + \sqrt{(-1 - m^2)} = n\, s - \sqrt{(-1 - n^2)}, \tag{89 e}$$

$$\left.\begin{array}{l} (1+s^2)\, m^2 - 2\, p\, s\, m + (1+p^2) = 0 \\ (1+s^2)\, n^2 - 2\, p\, s\, n + (1+p^2) = 0, \end{array}\right\} \tag{90 e}$$

$$m + n = \frac{2\, p\, s}{1+s^2}, \quad m\, n = \frac{1+p^2}{1+s^2}, \tag{91 e}$$

$$D_n p = m\, D_n\, s, \quad D_m\, p = n\, D_m\, s, \tag{92 e}$$

Integration of equation of surface of minimum extent.

$$D_n \varphi' = (m D_p \varphi' + D_s \varphi') D_n s, \qquad (93\text{ e})$$

$$\begin{aligned} D^2_{m.n} \varphi' &= [m n D^2_p \varphi' + (m+n) D^2_{s.p} \varphi' + D^2_s \varphi'] D_m s . D_n s \\ &\quad + D_p \varphi' D_m s + (n D_p \varphi' + D_s \varphi') D^2_{m.n} s \\ &= -[(1+p^2) D^2_p \varphi' + 2 p s D^2_{s.p} \varphi' + (1+s^2) D^2_s \varphi'] \\ &\quad \frac{1+s^2}{4\sqrt{(-1-m^2)}\sqrt{(-1-n^2)}} - \frac{(2 p D_p \varphi' + 2 s D_s \varphi')(1+s^2)}{4\sqrt{(-1-m^2)}\sqrt{(-1-n^2)}} \\ &= 0. \end{aligned} \qquad (94\text{ e})$$

Hence we find by integration,

$$\varphi' = F.m - F.m_\tau + f.n, \qquad (95\text{ e})$$

in which $f.n$ is the function which φ' becomes when m becomes m_τ, and $D_m f.m$ is the function which $D_m \varphi'$ becomes for all values of n. The equations (75 e and 83 e) give the values of ψ, t and x in terms of m and n. But it may be observed that t is the same as φ'.

CHAPTER XIII.

PARTICULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS.

229. In addition to the integral of a differential equation, there are *particular solutions*, which are not included in the general forms of the integral.

230. *Problem. To find a particular solution of a differential equation of the first order between two variables.*

Solution. Let x and t be the variables, and let

$$R = 0 \tag{96 e}$$

be the equation, and let

$$X = 0 \tag{97 e}$$

be the required solution which is supposed not to be included in the general form of the integral represented by

$$V = 0. \tag{98 e}$$

If, however, (97 e) were a case of (98 e) corresponding to the value x_{τ}' of the arbitrary constant x_{τ} involved in (98 e), and if we put

$$h = x_{\tau} - x_{\tau}', \tag{99 e}$$

the difference between the values of x derived from (97 e and 98 e) must vanish with h, so that where h is an infinitesimal, if x' denotes the value of x derived from (98 e), and x its value from (99 e), we may put

$$x' - x = X' h^n, \tag{1 f}$$

in which X' is a function of x and t different from zero. If p is the value of $D_t\, x$ given by (97 e), and p' its value given by (98 e), we shall also have, when $x' - x$ is an infinitesimal,

$$p' - p = P' (x' - x)^m = P' X'^m h^{mn}, \qquad (2\text{ f})$$

in which P' is a function of x and t different from zero.

But the differential of (1 f) gives

$$p' - p = D_t X'. h^n. \qquad (3\text{ f})$$

Whence we must have, if (97 e) is a case of (98 e),

$$P' X'^m h^{mn} = D_t X'. h^n. \qquad (4\text{ f})$$

But if m is less than unity, h^{mn} will be infinitely greater than h^n, and the equation (4 f) becomes

$$P' X'^m = 0, \qquad (5\text{ f})$$

which is impossible, so that in this case (4 f) cannot be satisfied, and (97 e) is not a case of (98 e), and is consequently a *particular solution.*

If m had been unity, (4 f) would have been reduced to

$$P' X' = D_t X', \qquad (6\text{ f})$$

which is easily satisfied.

If m were greater than unity, (4 f) becomes

$$D_t X' = 0, \quad X' = \text{constant}, \qquad (7\text{ f})$$

so that a particular solution is only indicated by the condition that m is less than unity.

The differential of (2 f) gives

$$D_{x'}\, p' = m\, P' (x' - x)^{m-1}, \qquad (8\text{ f})$$

which, when x' differs infinitely little from x and m is less than unity, gives

$$D_x\, p = \infty = \tfrac{1}{0}; \qquad (9\text{ f})$$

that is, $D_x\, p$ is a fraction whose denominator is zero.

The differentiation of (96 e) relatively to x gives, by substituting p for $D_t\, x$,

$$D_p\, R \,.\, D_x p + D_x\, R = 0, \tag{10 f}$$

or

$$D_x p = -\frac{D_x\, R}{D_p\, R}. \tag{11 f}$$

Whence by (9 f),

$$D_p\, R = 0, \tag{12 f}$$

provided the numerator cannot become infinity, which will be the case when (96 e) is free from radicals and fractions. This equation (12 f) corresponds to the particular solution, and leads to the particular solution by the elimination of p between it and the given equation (96 e).

231. *Corollary.* A similar method of finding particular solutions may be extended to other differential equations.

232. Examples.

1. Find the particular solution of the equation

$$t + x\, D_t\, x = d_t\, x \sqrt{(x^2 + t^2 - a^2)}. \tag{13 f}$$

Solution. This equation, freed from radicals, becomes

$$(t + x\, p)^2 = p^2\, (x^2 + t^2 - a^2),$$

whence (12 f) becomes

$$x\, (t + x\, p) = p\, (x^2 + t^2 - a^2).$$

The elimination of p gives for equation

$$(x^2 - a^2)\, (x^2 + t^2 - a^2) = 0,$$

of which the factor

$$x^2 + t^2 - a^2 = 0 \tag{14 f}$$

is the particular solution.

Particular solutions.

2. Find the particular solution of the equation

$$x - t\, D_t\, x + P, \qquad (15\text{ f})$$

in which P is a given function of $D_t\, x$.

Ans. It is the equation obtained by the elimination of p between the equation

$$\left.\begin{aligned} x &= t\, p + P' \\ \text{and} \qquad t + D_p\, P' &= 0, \end{aligned}\right\} \quad (16\text{ f})$$

in which P' is the value of P obtained by the substitution of p for $D_t\, x$.

THE END.

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